

Introduction to reflection groups and Coxeter groups

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References

- [Hum90] James E. Humphreys. *Reflection groups and Coxeter groups*. Vol. 29. Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 1990, pp. xii+204. ISBN: 0-521-37510-X. DOI: 10.1017/CB09780511623646. URL: <https://doi.org/10.1017/CB09780511623646>.

1 Reflections and reflection groups

2 Root systems

3 Root bases and Coxeter diagrams

4 Coxeter groups

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2 Root systems

3 Root bases and Coxeter diagrams

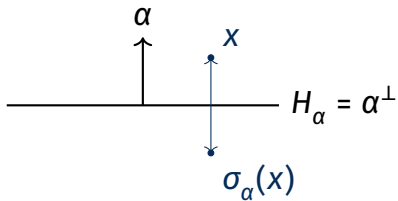
4 Coxeter groups

Notation. Throughout, V is a Euclidean vector space: a finite-dimensional real vector space with a positive-definite symmetric bilinear form $\cdot$. We write $\|v\| := \sqrt{v \cdot v}$ for the induced norm on V .

Reflections and roots

Definition (Reflection)

- (a) Let $\alpha \in V \setminus \{0\}$ and $H_\alpha := \alpha^\perp := \{v \in V \mid v \cdot \alpha = 0\}$. We denote by σ_α the *reflection along H_α* , and we call H_α its *reflection hyperplane*.
- (b) $s \in \text{GL}(V)$ is a *reflection* if $s = \sigma_\alpha$ for some $\alpha \in V \setminus \{0\}$. Any such α is called a *root of s* . If $\|\alpha\| = 1$, we call α a *normalised root of s* .



Remark.

- $\sigma_\alpha^2 = \text{id}$.
- The roots of σ_α are the elements of $\mathbb{R}_{\neq 0}\alpha$.

Reflection groups

Definition (Reflection group)

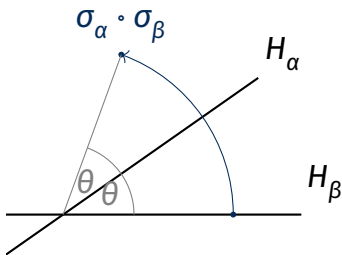
A *finite reflection group (in V)* is a finite group $W \leq \mathrm{GL}(V)$ for which there exists $R \subseteq V \setminus \{0\}$ such that

$$W = \mathcal{W}(R) := \langle \sigma_\alpha \mid \alpha \in R \rangle.$$

Remark.

- Every element of $\mathcal{W}(R)$ is a product of reflections, but not necessarily a reflection itself.
- The set R is not uniquely determined by $\mathcal{W}(R)$: For example, $\mathcal{W}(\{v\}) = \mathcal{W}(\{v, -v\})$ for all $v \in V \setminus \{0\}$.

Products of reflections



The product $\sigma_\alpha \circ \sigma_\beta$ of two reflections at an angle $\theta \in [0, \pi]$ is a rotation by the angle 2θ .

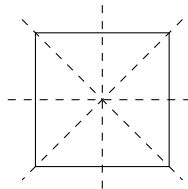
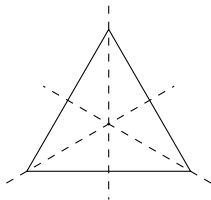
Thus if $\theta = \pi/m$, $m \in \mathbb{N}_{\geq 2}$, then we have the so-called *braid relation*:

$$(\sigma_\alpha \sigma_\beta)^m = \text{id}_V.$$

More generally, $\sigma_\alpha \sigma_\beta$ has finite order if and only if θ is a rational multiple of π .

Conclusion. A “random” choice of reflections usually generates an infinite group. Hence finite reflections group are special, which makes it plausible that they can be classified.

Symmetry groups of regular polygons/polyhedra



The symmetry groups of the regular triangle and of the square are generated by the reflections along the dashed lines. Hence they are finite reflection groups.

Classification in dimension 2. The finite reflection groups in dimension 2 are precisely the symmetry groups of the regular m -gons for $m \geq 3$.

Classification in dimension 3. The finite reflection groups in dimension 3 are precisely the symmetry groups of the Platonic solids.

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2 **Root systems**

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The root system of a reflection group

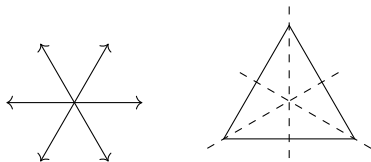
Definition (Root system of W)

Let W be a finite reflection group. Its (*normalised*) *root system* is

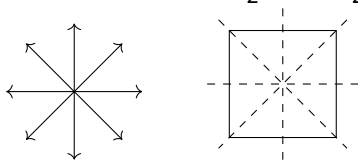
$$\Phi_{\text{norm}}(W) := \{ \alpha \in V \mid \|\alpha\| = 1 \text{ and } \sigma_{\alpha} \in W \}.$$

Equivalently, $\Phi_{\text{norm}}(W)$ is the *maximal* set R of unit vectors with $W = \mathcal{W}(R)$.

Example. The root systems of the previous examples are called A_2 and B_2 :



A_2



B_2

Abstract root systems

Definition (Abstract root system)

An (*abstract*) *root system in V* is a finite subset $\Phi \subseteq V \setminus \{0\}$ satisfying some axioms, most importantly:

- Φ is closed under its own reflections: $\sigma_\alpha(\Phi) = \Phi$ for all $\alpha \in \Phi$.

The group $\mathcal{W}(\Phi)$ is called the *Weyl group of Φ* .

Remark. We have a bijective correspondence between abstract root systems in V and finite reflection groups in V . It is given by

$$W \mapsto \Phi_{\text{norm}}(W) \quad \text{and} \quad \Phi \mapsto \mathcal{W}(\Phi).$$

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Root bases

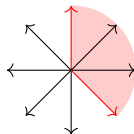
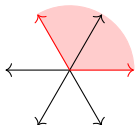
Convention. In this section, we fix a root system Φ in V and we assume that $\langle \Phi \rangle_{\mathbb{R}} = V$.

Definition (Root base)

A *root base of Φ* is a vector space basis $\Delta \subseteq \Phi$ of V such that for every

$$\alpha = \sum_{\delta \in \Delta} c_{\delta} \delta \in \Phi,$$

we have $c_{\delta} \geq 0$ for all $\delta \in \Delta$ or $c_{\delta} \leq 0$ for all $\delta \in \Delta$. The reflections σ_{δ} for $\delta \in \Delta$ are called *simple reflections (with respect to Δ)*.



Some properties of root bases

Theorem

- (a) There exists a root base of Φ .
- (b) $\mathcal{W}(\Phi)$ acts on the set of root bases of Φ .
- (c) This action is *simply transitive*: for root bases Δ_1, Δ_2 , there is a unique $w \in \mathcal{W}(\Phi)$ with $w(\Delta_1) = \Delta_2$.

Theorem

For any root base Δ of Φ , the simple reflections $(\sigma_\delta)_{\delta \in \Delta}$ generate $\mathcal{W}(\Phi)$.

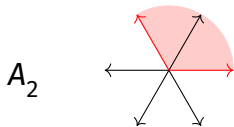
Coxeter matrices

Definition (Coxeter matrix of a root base)

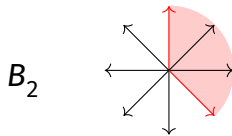
For a root base Δ with set $S := \{\sigma_\delta \mid \delta \in \Delta\}$ of simple reflections, the *Coxeter matrix of Δ* is the matrix $M(\Delta) := (\text{ord}(st))_{s,t \in S}$ over $\mathbb{Z}$.

Remark. Any Coxeter matrix is symmetric with ones on the diagonal.

Example. For $\alpha, \beta \in \Delta$, the order $\text{ord}(\sigma_\alpha \sigma_\beta)$ coincides with the number of roots in the cone spanned by α and β :



$$M(\Delta) = \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}$$



$$M(\Delta) = \begin{pmatrix} 1 & 4 \\ 4 & 1 \end{pmatrix}$$

The Coxeter matrix determines the Weyl group

Lemma

The Coxeter matrix of Φ “does not depend on the choice of the root base Δ ”.

Theorem (Coxeter matrix determines abstract isomorphism type)

Let $(m_{st})_{s,t \in S}$ be the Coxeter matrix of Φ . Then $\mathcal{W}(\Phi)$ has the presentation

$$\langle S \mid (st)^{m_{st}} = 1 \text{ for } s, t \in S \rangle.$$

Warning. Two Weyl groups that are isomorphic as abstract groups need not be isomorphic as reflection groups!

Coxeter diagrams

Definition (Coxeter diagram)

Let $M = (m_{st})_{s,t \in S}$ be a Coxeter matrix over a set S . The *Coxeter diagram of M* is the labelled graph $\Gamma(M)$ with vertex set S whose edges are given as follows:

- Two vertices $s, t \in S$ are joined by an edge if and only if $m_{st} \geq 3$,
- this edge is labelled by m_{st} .
- When drawing Coxeter diagrams, we use the convention that edges with label $m_{st} = 3$ are drawn as unlabelled edges and edges with label $m_{st} = 4$ are drawn as double edges without a label.

Classification of (irreducible) Coxeter diagrams

Name	Coxeter diagram
A_n	
B_n	
D_n	
E_6	
E_7	
E_8	

Name	Coxeter diagram
F_4	
G_2	
H_2	
H_3	
H_4	
$I_2(m)$	

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Coxeter groups

Definition (Coxeter matrix)

Let S be any set. A *Coxeter matrix over S* is a matrix $M = (m_{st})_{s,t \in S}$ with entries in $\mathbb{N}_{\geq 1} \cup \{\infty\}$ such that the following conditions are satisfied:

- $m_{ss} = 1$ for all $s \in S$.
- $m_{st} = m_{ts} \geq 2$ for all $s \neq t \in S$.

Definition (Coxeter group)

For a Coxeter matrix $M = (m_{st})_{s,t \in S}$ over S , the *Coxeter group of type M* is the abstract group given by the presentation

$$\mathcal{W}(M) := \langle S \mid (st)^{m_{st}} = 1 \text{ for } s, t \in S \text{ with } m_{st} \neq \infty \rangle.$$

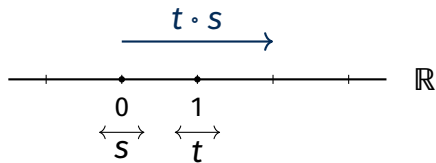
Example: Infinite dihedral group

By allowing reflections along hyperplanes that do *not* go through the origin, we obtain *affine reflection groups*.

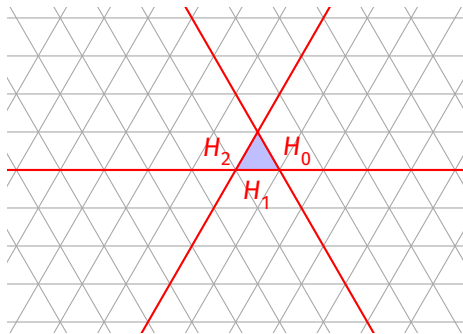
Example. The *infinite dihedral group*

$$\mathcal{W}(I_2(\infty)) = D_\infty = \langle s, t \mid s^2 = t^2 = 1 \rangle$$

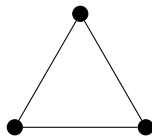
is a Coxeter group of type $\bullet \overset{\infty}{\text{---}} \bullet$.



The affine group of type $\tilde{A}_2$



Example. The reflections along H_0, H_1, H_2 generate the *reflection group of type $\tilde{A}_2$* , a subgroup of $\text{Aff}(\mathbb{R}^2)$. It is of type



The stabiliser of any lattice point is a reflection subgroup of type A_2 .

Summary

- (a) A *finite reflection group in V* is a finite subgroup of $GL(V)$ which is generated by reflections.
- (b) These objects are in bijective correspondence with *root systems in V* .
- (c) Each root system has a *Coxeter matrix*, which is described succinctly by its *Coxeter diagram*.
- (d) Each finite reflection group has a *finite presentation* involving the entries of its Coxeter matrix.
- (e) Allowing arbitrary (abstract) Coxeter matrices in this presentation leads to the (or rather, one) definition of *Coxeter groups*.