

Introduction to reflection groups and Coxeter groups

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Abstract

This talk introduces the notion of finite (real) reflection groups (that is, finite Coxeter groups). We will cover their relationship to finite root systems and Coxeter diagrams, study some examples and give an overview of their classification. At the end of the talk, we will also briefly discuss non-finite Coxeter groups.

Remark. LLMs have been used in the creation of Figures 1, 2, 7 and 8, for structuring the $\text{\LaTeX}$ code of this document and for finding typos, but not in the writing of the text.

Convention

Throughout this talk, we fix a Euclidean vector space V . That is, V is a finite-dimensional real vector space equipped with a positive-definite symmetric bilinear form $\cdot$. We denote by $\|\cdot\| : V \rightarrow \mathbb{R}_{\geq 0}, v \mapsto \sqrt{v \cdot v}$ the corresponding norm. Further, homomorphisms of vector spaces always act from the left.

1 Reflections

1.1 Definition (Reflection)

- (a) For each $\alpha \in V \setminus \{0\}$, the *reflection along $\alpha^\perp$* is the unique linear map $\sigma_\alpha : V \rightarrow V$ which fixes $\alpha^\perp := \{v \in V \mid v \cdot \alpha = 0\}$ and sends α to $-\alpha$. Its *reflection hyperplane* is $H_\alpha := H_{\sigma_\alpha} := \alpha^\perp = \text{Fix}(\sigma_\alpha)$.
- (b) An automorphism $s \in \text{GL}(V)$ is called a *reflection* if there exists $\alpha \in V \setminus \{0\}$ with $s = \sigma_\alpha$, and any such α is called a *root of s* . If in addition $\|\alpha\| = 1$, then α is called a *normalised root of s* .

1.2 Remark.

- (1) If s is a reflection in V with root α , then $\{\lambda\alpha \mid 0 \neq \lambda \in \mathbb{R}\}$ is the set of all roots of s . Exactly two of these roots are normalised, and they are negatives of each other.
- (2) For all $\alpha \in V \setminus \{0\}$, the map σ_α is the unique reflection in V for which α is a root.
- (3) Every reflection s is orthogonal: We have $s(v) \cdot s(w) = v \cdot w$ for all $v, w \in V$.

1.3 Remark. Let s be a reflection in V . Then $s^2 = \text{id}_V$, and the action of s on V can be depicted as in Figure 1. We conclude from this observation that the study of a single reflection is not a particularly deep

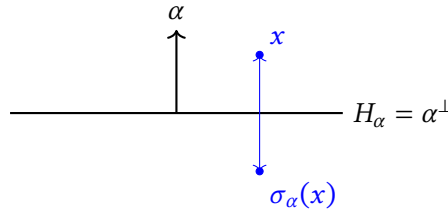


Figure 1: Graphical depiction of a reflection σ_α with root α and reflection hyperplane H_α .

subject. However, things become more interesting when we study the interplay of several reflections, which ultimately leads to the notion of reflection groups.

1.4 Remark (Products of two reflections). Assume that V is two-dimensional, let $\alpha, \beta \in V$ with $\|\alpha\| = 1 = \|\beta\|$ and let $\theta \in [0, \pi]$ denote the angle between α and β (that is, $\alpha \cdot \beta = \cos \theta$). Then a short matrix computation, which we skip, shows that the product $\sigma_\alpha \circ \sigma_\beta$ is a rotation by the angle 2θ (see Figure 2). In particular, $\sigma_\alpha \sigma_\beta$ has finite order if and only if θ is a rational multiple of π : Explicitly, if $\theta = \pi/m$ for some $m \in \mathbb{N}_{\geq 2}$, then $\sigma_\alpha \sigma_\beta$ has order exactly m . In particular, we have the relation

$$(\sigma_\alpha \sigma_\beta)^m = \text{id}_V,$$

which is called a *braid relation*¹. We will see in Theorem 4.14 that these relations together with the relations $\sigma_\alpha^2 = \text{id}_V$ are sufficient to describe finite reflection groups in terms of a finite presentation.

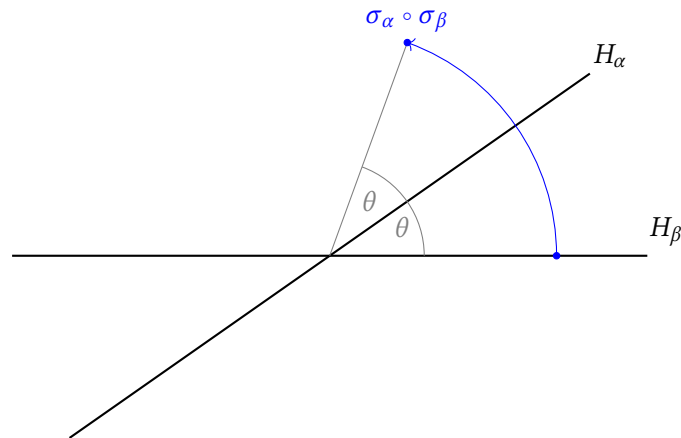


Figure 2: The product of two reflections is a rotation.

2 Reflection groups

2.1 Definition (The reflection group generated by a set of roots)

(a) Let R be a subset of $V \setminus \{0\}$. We define

$$\mathcal{W}(R) := \langle \sigma_\alpha \mid \alpha \in R \rangle \leq \text{GL}(V),$$

the group generated by all reflections σ_α for $\alpha \in R$.

¹More precisely, the relation $\sigma_\alpha \sigma_\beta \sigma_\alpha \cdots = \sigma_\beta \sigma_\alpha \sigma_\beta \cdots$, where both products have length $m/2$, is called a braid relation. Given the relation $\sigma_\alpha^2 = 1 = \sigma_\beta^2$, it is equivalent to the relation $(\sigma_\alpha \sigma_\beta)^m = 1$.

(b) A subgroup W of $\text{GL}(V)$ is called a *finite reflection group (in V)* if W is finite and there exists a set $R \subseteq V \setminus \{0\}$ with $W = \mathcal{W}(R)$.

2.2 Remark. (1) Every element of a finite reflection group is a product of reflections, but it need not be a reflection itself. See, for example, Remark 1.4.

(2) Note that a reflection group in V is always equipped with a natural action on V . Alternatively, we may define a reflection group in V as a pair (W, i) where W is an abstract group and $i : W \rightarrow \text{GL}(V)$ is an injective homomorphism whose image is a reflection group in our sense. It may happen that a given abstract group has multiple rather different realisations as a reflection group (see also Remark 4.15). A more careful study of this phenomenon would require us to define the notion of isomorphism for reflection groups, which we omit. However, we point out that we can always ignore the subspace $R^\perp = \{v \in V \mid v \cdot \alpha = 0 \text{ for all } \alpha \in R\}$ on which $\mathcal{W}(R)$ acts trivially.

(3) For a given reflection group W , there are in general many sets R such that $W = \mathcal{W}(R)$. As a trivial example, $\mathcal{W}(\{v\}) = \mathcal{W}(\{v, -v\})$ for any $v \in V \setminus \{0\}$. To overcome this ambiguity, we will introduce the notion of the root systems (see also Remark 3.2).

2.3 Examples.

(1) The symmetry group of the regular triangle is generated by the reflections along the dashed lines in Figure 3a. Hence it is a finite reflection group, which we call *of type A_2* . It can be identified with the full permutation group of its vertices, that is, with $\text{Sym}(3)$. Similarly, the symmetry group of the regular tetrahedron is a finite reflection group that we call *of type A_3* , and it can be identified with $\text{Sym}(4)$. More generally, we have for all $n \in \mathbb{N}_{\geq 2}$ that the symmetry group of the n -dimensional simplex is a finite reflection group, which is called *of type A_n* and which can be identified with $\text{Sym}(n+1)$.

(2) The symmetry group of the square is generated by the reflections along the dashed lines in Figure 3b. Hence it is a finite reflection group as well, which we call *of type B_2* . Similarly, the symmetry group of the cube is a finite reflection group that we call *of type B_3* . It coincides with the symmetry group of the octahedron. Again, there are higher-dimensional analogues, which are called *hypercubes* or *n -cubes*.

(3) The symmetry groups of the remaining Platonic solids – that is, of the dodecahedron and the icosahedron – are also finite reflection groups. In fact, they coincide, and they are called *of type H_3* .

(4) Let $m \in \mathbb{N}_{\geq 3}$. Then the symmetry group of the regular m -gon is a finite reflection group that is called *of type $I_2(m)$* . It can be identified with the dihedral group of order $2m$, which is given by the presentation

$$\langle s, t \mid s^2 = t^2 = (st)^m = 1 \rangle.$$

For $m = 3$ and $m = 4$, we recover the symmetry groups of the regular triangle and square, respectively.

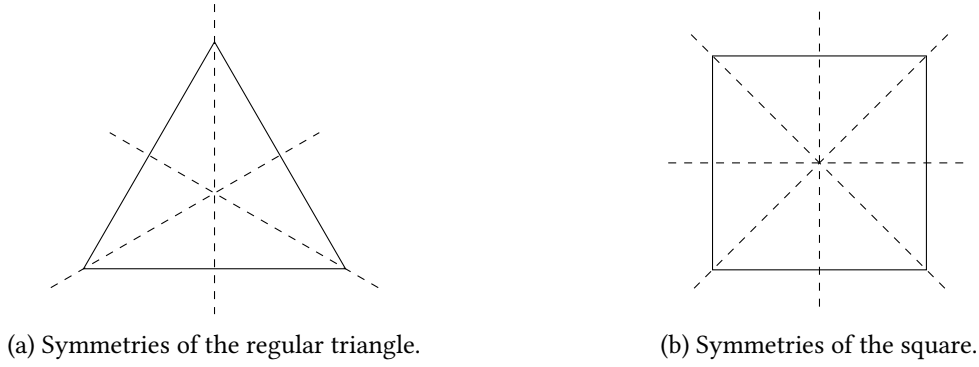


Figure 3: Symmetries of some regular polygons.

Platonic solid	Tetrahedron	Cube	Octahedron	Dodecahedron	Icosahedron
Number of faces	4	6	8	12	20
Symmetry group	A_3	B_3	B_3	H_3	H_3

Table 1: The symmetry groups of the Platonic solids.

3 Root systems

As a first step towards a classification of finite reflection groups, we introduce the notion of root systems. For any finite reflection group, we define an associated root system, and we will see that it completely determines the reflection group.

3.1 Definition (Root system of a reflection group)

Let W be a finite reflection group. The *(normalised) root system of W* is the set

$$\Phi_{\text{norm}}(W) := \{ \alpha \in V \mid \|\alpha\| = 1 \text{ and } \sigma_\alpha \in W \}.$$

3.2 Remark. In other words, the root system of a finite reflection group W is the maximal set R such that R consists of unit vectors and $W = \mathcal{W}(R)$.

3.3 Examples. The root systems of the reflection groups of the regular triangle and of the square look just like the dashed hyperplane arrangements in Figure 3: See Figure 4. While we cannot (easily) draw pictures of their higher-rank analogues, they are easy to describe algebraically: They are given inside the Euclidean space $V := \mathbb{R}^n$ in terms of the standard basis $(e_1, \dots, e_n)$ by

$$A_{n-1} := \left\{ \frac{1}{\sqrt{2}}(e_i - e_j) \mid 1 \leq i \neq j \leq n \right\} \quad \text{and} \\ B_n := \{ \pm e_i \mid 1 \leq i \leq n \} \cup \left\{ \frac{1}{\sqrt{2}}(\varepsilon_1 e_i + \varepsilon_2 e_j) \mid \varepsilon_1, \varepsilon_2 \in \{\pm 1\}, 1 \leq i < j \leq n \right\}.$$

Another example of a root system is given by

$$D_n := \left\{ \frac{1}{\sqrt{2}}(\varepsilon_1 e_i + \varepsilon_2 e_j) \mid \varepsilon_1, \varepsilon_2 \in \{\pm 1\}, 1 \leq i < j \leq n \right\} \subseteq V.$$



Figure 4: Examples of root systems in two-dimensional space.

3.4 Definition (Abstract root system)

A subset Φ of $V \setminus \{0\}$ is called a **root system** in V if the following hold:

- (a) Φ is finite.
- (b) Φ is **normalised**: For all $\alpha \in \Phi$, we have $\|\alpha\| = 1$.
- (c) Φ is **reduced**: For all $\alpha \in \Phi$, we have $\Phi \cap \mathbb{R}\alpha = \{\alpha, -\alpha\}$.
- (d) Φ is invariant under all reflections that it defines: $\sigma_\alpha(\Phi) = \Phi$ for all $\alpha \in \Phi$.

For a root system Φ , the group $\mathcal{W}(\Phi)$ is called the **Weyl group of Φ** .

3.5 Remark. Let Φ be a root system. By Axiom 3.4 (d), $\mathcal{W}(\Phi)$ acts on the finite set Φ . The action of $\mathcal{W}(\Phi)$ on $\langle \Phi \rangle_{\mathbb{R}}$ is determined by its action on Φ , and its action on the orthogonal complement of $\langle \Phi \rangle_{\mathbb{R}}$ is trivial. It follows that the action of $\mathcal{W}(\Phi)$ on Φ is faithful. Thus $\mathcal{W}(\Phi)$ is isomorphic to a subgroup of $\text{Sym}(|\Phi|)$, and hence finite. We conclude that Weyl groups of root systems are finite reflection groups.

3.6 Theorem (Correspondence between finite reflection groups and systems)

- (a) Let W be a finite reflection group (in V). Then $\Phi_{\text{norm}}(W)$ is a root system (in V).
- (b) Let Φ be a root system (in V). Then $\mathcal{W}(\Phi)$ is a finite reflection group (in V).
- (c) The assignments $\Phi \mapsto \mathcal{W}(\Phi)$ and $W \mapsto \Phi_{\text{norm}}(W)$ are mutually inversion bijections between the set of root systems in V and the set of finite reflection groups in V .

3.7 Remark. Theorem 3.6 reduces the classification of finite reflection groups in V to the classification of root systems in V . Note, however, that it makes no assertion about “isomorphic” reflection groups corresponding to “isomorphic” root systems. Indeed, we did not define the notion of isomorphism for these objects (see also Remark 2.2 (2)).

4 Root bases and Coxeter diagrams

4.1 Convention for this section

We fix a root system Φ in V .

In this section, we introduce the notion of Coxeter diagrams, which classify root systems “up to isomorphism”. This is the second step in the classification of finite reflection groups.

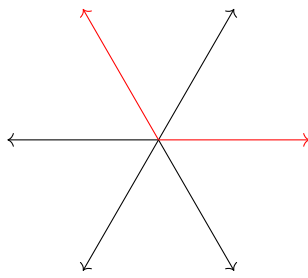
4.2 Definition (Root base)

A **root base** of Φ is a linearly independent subset Δ of Φ such that

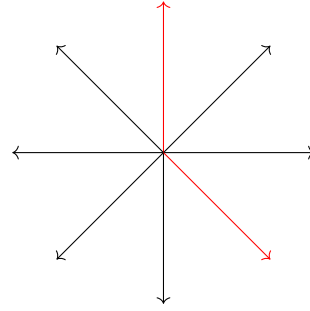
$$\Phi \subseteq \left\{ \sum_{\delta \in \Delta} c_\delta \delta \mid c_\delta \in \mathbb{R}_{\geq 0} \text{ for all } \delta \in \Delta \right\} \cup \left\{ \sum_{\delta \in \Delta} c_\delta \delta \mid c_\delta \in \mathbb{R}_{\leq 0} \text{ for all } \delta \in \Delta \right\}.$$

If the root base Δ is clear from the context, its elements are also called **simple roots** and the corresponding reflections σ_β for $\beta \in \Delta$, are called **simple reflections**.

4.3 Remark. The angle between two simple roots is always “as obtuse as possible”. It is possible to make this statement more precise, but we omit the details.



(a) A root base in A_2 , drawn in red.



(b) A root base in B_2 , drawn in red.

Figure 5: Examples of root systems in two-dimensional space. The corresponding reflection groups can be identified with the symmetry group of the blue polygon.

4.4 Theorem (Existence and uniqueness of root bases, [Hum90, 1.3, 1.4, 1.8])

- (a) There exists a root base of Φ .
- (b) The Weyl group $\mathcal{W}(\Phi)$ acts on the set of root bases: If Δ is a root base of Φ and $w \in \mathcal{W}(\Phi)$, then $w(\Delta)$ is a root base of Φ .
- (c) The action of the Weyl group $\mathcal{W}(\Phi)$ on the set of root bases of Φ is simply transitive: If Δ_1, Δ_2 are two root bases of Φ , then there exists a unique $w \in \mathcal{W}(\Phi)$ with $w(\Delta_1) = \Delta_2$.

4.5 Theorem (Generation by simple reflections, [Hum90, 1.5])

Let Δ be a root base in Φ . Then the simple reflections $(\sigma_\delta)_{\delta \in \Delta}$ generate $\mathcal{W}(\Phi)$.

Any root base has an attached combinatorial invariant, which is called its Coxeter matrix.

4.6 Definition (Coxeter matrix)

Let S be a set. A **Coxeter matrix over S** is a matrix $M = (m_{st})_{s,t \in S}$ with entries in $\mathbb{N}_{\geq 1} \cup \{\infty\}$ such that the following conditions are satisfied:

- (a) $m_{ss} = 1$ for all $s \in S$.
- (b) $m_{st} = m_{ts} \geq 2$ for all $s \neq t \in S$.

4.7 Definition (Coxeter matrix of a root base)

Let Δ be a root base of Φ and let S denote the corresponding set of simple reflections. The *Coxeter matrix of Δ* is the Coxeter matrix $M(\Delta) := (\text{ord}(st))_{s,t \in S}$ over S where $\text{ord}(st) \in \mathbb{N}$ is the order of st in the group $\mathcal{W}(\Phi)$.

4.8 Remark. Let Δ be a root base of Φ and denote its Coxeter matrix by $(m_{st})_{s,t \in S}$. This is indeed a Coxeter matrix:

- (1) For all $s \in S$, we have $m_{ss} = \text{ord}(s^2) = \text{ord}(1) = 1$.
- (2) For all $s, t \in S$, we have $(st)^{-1} = t^{-1}s^{-1} = ts$ and hence $m_{st} = \text{ord}(st) = \text{ord}(ts) = m_{ts}$.
- (3) For all $s \neq t \in S$, we have $st \neq 1$ and hence $m_{st} = \text{ord}(st) \geq 2$.

4.9 Remark. Not all Coxeter matrices arise from finite reflection groups: For example, the Coxeter matrix of a finite reflection group never contains an entry ∞ .

4.10 Lemma

Let Δ_1, Δ_2 be two root bases of Φ , let S_1, S_2 denote the corresponding sets of reflections and let $(m_{st})_{s,t \in S_1}, (m_{st})_{s,t \in S_2}$ denote the corresponding Coxeter matrices. Then there exists a bijection $f : S_1 \rightarrow S_2$ such that $m_{f(s)f(t)} = m_{st}$ for all $s, t \in S_1$.

4.11 Remark. By Lemma 4.10, the Coxeter matrices of all root bases of Φ are the same up to re-ordering the rows and columns. Hence we also speak of the *Coxeter matrix of Φ* . If W is a finite reflection group, the Coxeter matrix of $\Phi_{\text{norm}}(W)$ is also called the *Coxeter matrix of W* .

4.12 Definition (Coxeter diagram)

Let S be a set and let $M = (m_{st})_{s,t \in S}$ be a Coxeter matrix over S . The *Coxeter diagram of M* is the graph $\Gamma(M)$ with vertex set S whose edges and labels are given as follows: Two vertices $s, t \in S$ are joined by an edge if and only if $m_{st} \geq 3$, and this edge is labelled by m_{st} . When drawing Coxeter diagrams, we use the convention that edges with label $m_{st} = 3$ are drawn as unlabelled edges.

4.13 Remark. Let Δ be a root base of Φ and let $\alpha, \beta \in \Delta$. Then the order $\text{ord}(\sigma_\alpha \sigma_\beta)$ coincides with the number of roots in the “closed interval from α to β ”, that is, the number of roots “lying between” α and β (including α and β themselves).



Figure 6: The root systems of type A_2 and B_2 with a choice of root base Δ (in red) and the corresponding Coxeter matrix $M(\Delta)$.

4.14 Theorem (Presentation of finite reflection groups, [Hum90, 1.9])

Let Δ be a root base of Φ , let S be the associated set of simple reflections and denote the corresponding Coxeter matrix by $(m_{st})_{s,t \in S}$. Then the natural map $S \rightarrow \mathcal{W}(\Phi)$ induces an isomorphism between $\mathcal{W}(\Phi)$ and the finitely presented group

$$\langle S \mid (st)^{m_{st}} = 1 \text{ for } s, t \in S \rangle.$$

4.15 Remark. We conclude from Theorem 4.14 that the isomorphism type of a finite reflection group, regarded as an abstract group, is determined by its Coxeter diagram. However, distinct Coxeter diagrams may lead to isomorphic abstract groups. See also Remark 2.2 (2).

4.16 Definition (Irreducible Coxeter diagram)

A Coxeter diagram is called *irreducible* if it is connected as a graph. The *irreducible components* of a Coxeter diagram are its connected components.

4.17 Theorem (Classification of finite reflection groups)

- (a) Let W be a finite reflection group and let Φ denote its root system. Then the irreducible components of the Coxeter diagram of Φ are in the list in Table 2.
- (b) For each of the Coxeter diagrams Π in Table 2, there exist a Euclidean space V' and a finite reflection group in V' whose Coxeter diagram is Π .
- (c) Two finite reflection groups are “isomorphic as reflection groups”^a if and only if their Coxeter diagrams coincide.

^aIn the sense of Remark 2.2 (2), which we did not make precise.

5 Coxeter groups

Theorem 4.14 motivates the following definition of Coxeter groups, which generalise finite reflection groups.

5.1 Definition (Coxeter group)

Let S be a set and let $M = (m_{st})_{s,t \in S}$ be a Coxeter matrix over S . The *Coxeter group of type M* is the group given by the presentation

$$\mathcal{W}(M) := \langle S \mid (st)^{m_{st}} = 1 \text{ for } s, t \in S \text{ with } m_{st} \neq \infty \rangle.$$

5.2 Examples (Affine Coxeter groups). The easiest examples of infinite Coxeter groups are the affine ones. They are obtained by allowing reflections that are not linear but only affine: In other words, we allow reflection hyperplanes that do not contain the zero vector. These groups can be identified with subgroups of $\text{Aff}(V)$, the group of affine transformations of V .

- (1) The *infinite dihedral group*

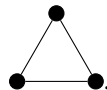
$$\mathcal{W}(I_2(\infty)) = D_\infty = \langle s, t \mid s^2 = t^2 = 1 \rangle$$

Name	Coxeter diagram
A_n	
B_n	
D_n	
E_6	
E_7	
E_8	
F_4	
G_2	
H_2	
H_3	
H_4	
$I_2(m)$	

Table 2: The Coxeter diagrams of finite reflection groups. The subscript of the type stands for the number of vertices of the diagram.

is a Coxeter group of type $\bullet \overset{\infty}{\text{---}} \bullet$. We can realise it inside $\text{Aff}(\mathbb{R}^1)$ by identifying s and t with the reflections along the hyperplanes (that is, points) $\{0\}$ and $\{1\}$. Then $t \circ s(x) = x + 2$ for all $x \in \mathbb{R}$, so $t \circ s$ has infinite order. See Figure 7.

- (2) The *reflection group of type $\tilde{A}_2$* is the subgroup of $\text{Aff}(\mathbb{R}^2)$ that is generated by the reflections along all hyperplanes in Figure 8. In fact, it is already generated by the simple reflections along H_0, H_1, H_2 . Its Coxeter diagram is



For every lattice point p (that is, every point on the intersection of three hyperplanes), the stabiliser of p is a reflection subgroup of type A_2 .

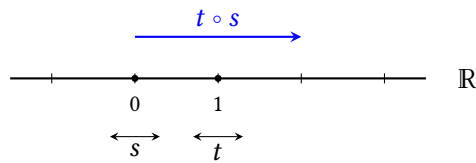


Figure 7: The infinite dihedral group as a subgroup of $\text{Aff}(\mathbb{R}^1)$.

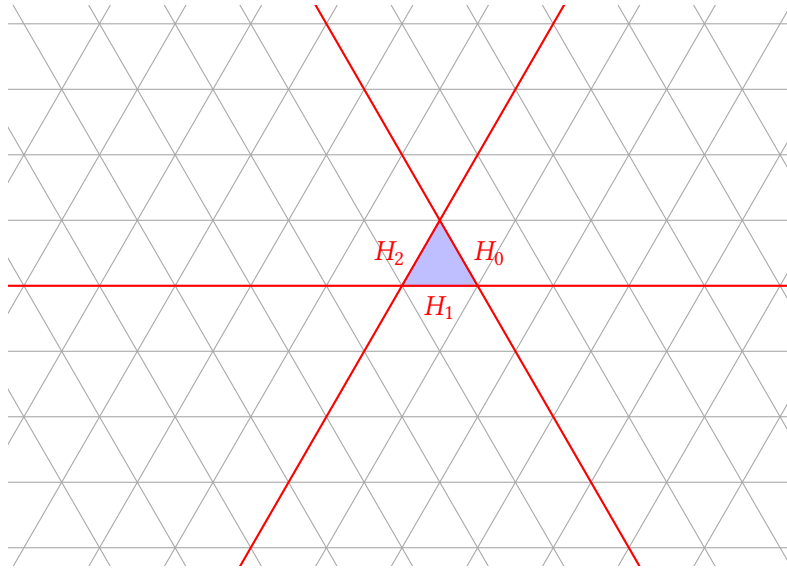


Figure 8: A tiling of the plane by regular triangles. The reflections along the three hyperplanes H_0 , H_1 , H_2 generate the affine Coxeter group $\mathcal{W}(\tilde{A}_2)$.

References

- [Hum90] James E. Humphreys. *Reflection groups and Coxeter groups*, volume 29 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1990.