HALL–LITTLEWOOD POLYNOMIALS, AFFINE SCHUBERT SERIES, AND LATTICE ENUMERATION

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Abstract. We introduce multivariate rational generating series called Hall–Littlewood–Schubert (HLS_n) series. They are defined in terms of polynomials related to Hall–Littlewood polynomials and semistandard Young tableaux. We show that HLS_n series provide solutions to a range of enumeration problems upon judicious substitutions of their variables. These include the problem to enumerate sublattices of a p-adic lattice according to the elementary divisor types of their intersections with the members of a complete flag of reference in the ambient lattice. This is an affine analog of the stratification of Grassmannians by Schubert varieties. Other substitutions of HLS_n series yield new formulae for Hecke series and p-adic integrals associated with symplectic p-adic groups, and combinatorially defined quiver representation zeta functions. HLS_n series are q-analogs of Hilbert series of Stanley–Reisner rings associated with posets arising from parabolic quotients of Coxeter groups of type B with the Bruhat order. Special values of coarsened HLS_n series yield analogs of the classical Littlewood identity for the generating functions of Schur polynomials.

Contents

Introduction

We offer a unifying framework for a wide variety of counting problems from geometry, number theory, and algebra. To this end we introduce *Hall–Littlewood–Schubert series* HLS_n; see Definition 1.2. These are multivariate rational generating functions defined as sums over semistandard Young tableaux (or just tableaux in the sequel), involving polynomials related to Hall–Littlewood polynomials. We

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show that they specialize, under judicious substitutions of their 2^n variables, to generating series solving various counting problems.

What makes each of these problems amenable to Hall–Littlewood–Schubert series is that they all factor over natural maps from the set of all finite-index sublattices of a fixed lattice of finite rank n to the set SSYT_n of tableaux with entries from $\{1, \ldots, n\}$. In each case, the key to reducing the respective counting problem to HLS_n is to compute the fibers of the relevant map. En route we discover connections with further classical objects of algebraic combinatorics, such as Dyck words, the Bruhat order, and Stanley–Reisner rings. Three such instantiations, all related to lattice enumeration, stand out.

(1) Let V be a module over a compact discrete valuation ring $\mathfrak{o}$, free of finite rank n, equipped with a complete flag of isolated submodules $\{0\} = V^{(0)} \subsetneq V^{(1)} \subsetneq V^{(2)} \subsetneq \cdots \subsetneq V^{(n)} = V$. The *affine Schubert series* $\mathrm{affS}^{\mathrm{in}}_{n,\mathfrak{o}}$ introduced in Definition 1.5 enumerates sublattices of finite index in V by the elementary divisors of their intersections with each of the lattices $V^{(i)}$. This may be seen as an affine analog of the classical concept of Schubert varieties, stratifying Grassmannians by the intersection dimensions with a fixed complete flag in the ambient vector space; see [7]. Theorem B asserts that HLS_n specializes to $\mathrm{affS}^{\mathrm{in}}_{n,\mathfrak{o}}$ under a monomial substitution of the variables. Theorem C is a similar result for the affine Schubert series $\mathrm{affS}^{\mathrm{pr}}_{n,\mathfrak{o}}$, enumerating lattices by the elementary divisors of their projections to, rather than intersections with, the members of a complete flag of reference.

(2) *Hecke series* play an important role in algebra and number theory. Theorem E shows that Hall–Littlewood–Schubert series HLS_n specialize to the Hecke series associated with groups of symplectic similitudes over local fields as studied by Macdonald [12, Ch. V]. This leads to new formulae for and new results about these classical series.

(3) *Quiver representation zeta functions* enumerate subrepresentations of integral quiver representations; see [10]. Specializations of Hall–Littlewood–Schubert series yield new and explicit formulae for these zeta functions associated with combinatorially defined quiver representations over compact discrete valuation rings.

Additional applications flow from the fact that HLS_n is a Y-analog of the Hilbert series of the Stanley–Reisner ring of a natural simplicial complex. This is the order complex $\Delta(\mathsf{T}_n)$ of the poset $\mathsf{T}_n = 2^{[n]}\backslash\{\varnothing\}$ equipped with the *tableaux order* introduced in Section 6.1. The poset T_n may be interpreted in terms of the Bruhat order on parabolic quotients of finite Coxeter groups of type B.

We state a general self-reciprocity result (Theorem A) for the Hall–Littlewood–Schubert series HLS_n upon inversion of their variables. Through the relevant variable substitutions, self-reciprocity is passed on to the generating series described above, vastly extending the scope of this well-studied symmetry phenomenon. Our proof of Theorem A is facilitated by interpreting HLS_n in terms of $\mathfrak{p}$-adic integrals. Conversely, we give pleasing formulae for well-studied $\mathfrak{p}$-adic integrals associated with symplectic $\mathfrak{p}$-adic groups in terms of Hall–Littlewood–Schubert series.

1. Main objects and main results

We defer precise definitions, even of standard objects pertaining to partitions, tableaux, Young diagrams, lattices, and flags, to Section 2. Throughout, let $n \in \mathbb{N}$.

1.1. Hall–Littlewood–Schubert series.

Write SSYT_n for the set of tableaux $T = (T_{ij})$ of degree n, i.e. with labels in $[n] := \{1, \ldots, n\}$. Write $T = (C_1, \ldots, C_\ell)$ to denote the columns of $T \in \mathrm{SSYT}_n$. For $i, j \in \mathbb{N}$ we define the ***leg set*** of T:

$$\mathrm{Leg}^{+}_{T}(i,j) = \begin{cases} C_j \cap [T_{ij},\ T_{i(j+1)}] & \text{if } T_{i(j+1)} \notin C_j, \\ \varnothing & \text{otherwise.} \end{cases}$$

We set $\mathscr{L}_T = \left\{(i,j) \in \mathbb{N}^2 \mid \mathrm{Leg}_T^+(i,j) \neq \varnothing\right\}$.

Definition 1.1. The ***leg polynomial*** associated with $T \in \mathrm{SSYT}_n$ is

$$\Phi_T(Y) = \prod_{(i,j)\in\mathscr{L}_T} \left(1 - Y^{\#\,\mathrm{Leg}_T^+(i,j)}\right) \in \mathbb{Z}[Y].$$

We introduce further $2^n - 1$ variables $\boldsymbol{X} = (X_C)_{\varnothing\neq C\subseteq[n]}$. We call a tableau ***reduced*** if its columns are pairwise distinct and write rSSYT_n for the finite (!) set of reduced tableaux of degree n.

Definition 1.2. The ***Hall–Littlewood–Schubert series*** is

$$\mathsf{HLS}_n\left(Y, \boldsymbol{X}\right) = \sum_{T\in\mathrm{rSSYT}_n} \Phi_T(Y) \prod_{C\in T} \frac{X_C}{1 - X_C} \in \mathbb{Z}[Y]\left(\boldsymbol{X}\right).$$

Remark 1.3. The leg polynomial Φ_T coincides with a known polynomial invariant of Gelfand–Tsetlin patterns, which are known to be in bijection with tableaux; see Lemma 6.11. For a partition λ, let $P_\lambda(\boldsymbol{x}; t)$ be the *Hall–Littlewood polynomial*. In Equation (6.7) we reproduce an expression, due to Feighin–Maklin, for $P_\lambda(\boldsymbol{x}; t)$ as a (finite) sum indexed by the tableaux $T \in \mathrm{SSYT}_n$ of shape λ, involving both the leg polynomials Φ_T and the weights of the tableaux. By recording the $2^n - 1$ possible label sets of columns of tableaux of degree n, the Hall–Littlewood–Schubert series keeps track of much finer information.

We note that leg sets index the cells contained in the leg of the (i,j)-cell for a suitable partition in Macdonald's terminology; see [12, p. 337].

We define the denominator polynomial

$$\mathsf{D}_n(\boldsymbol{X}) = \prod_{\varnothing\neq C\subseteq[n]} (1 - X_C) \in \mathbb{Z}[\boldsymbol{X}].$$

We then define the numerator polynomial $\mathsf{N}_n(Y, \boldsymbol{X}) \in \mathbb{Z}[Y, \boldsymbol{X}]$ via

$$\mathsf{HLS}_n(Y, \boldsymbol{X}) = \frac{\mathsf{N}_n(Y, \boldsymbol{X})}{\mathsf{D}_n(\boldsymbol{X})}. \tag{1.1}$$

Example 1.4 (HLS_n for $n \leqslant 3$)**.** Given subsets $I_1, I_2, \ldots \subset \mathbb{N}$ we write $\boldsymbol{X}_{I_1|I_2|\ldots} = X_{I_1}X_{I_2}\cdots$. We further simplify the subscripts by displaying only the sets' elements: for example, we write X_{13} instead of $X_{\{1,3\}}$. For $n \leqslant 3$, we find

$$\mathsf{N}_1(Y, \boldsymbol{X}) = 1, \quad \mathsf{N}_2(Y, \boldsymbol{X}) = 1 - Y\boldsymbol{X}_{1|2},$$

and

$$\begin{aligned}
\mathsf{N}_3(Y, \boldsymbol{X}) &= 1 - \boldsymbol{X}_{1|23}\\
&- Y\left(\boldsymbol{X}_{1|2} + \boldsymbol{X}_{1|3} + \boldsymbol{X}_{2|3} + \boldsymbol{X}_{2|13} + \boldsymbol{X}_{12|13} + \boldsymbol{X}_{12|23} + \boldsymbol{X}_{13|23} + \boldsymbol{X}_{2|13|23} + \boldsymbol{X}_{1|2|13|23}\right)\\
&+ Y\left(\boldsymbol{X}_{1|2|3} + \boldsymbol{X}_{1|2|13} + \boldsymbol{X}_{1|2|23} + \boldsymbol{X}_{1|3|23} + \boldsymbol{X}_{1|12|23} + \boldsymbol{X}_{1|13|23} + \boldsymbol{X}_{12|13|23}\right)\\
&+ Y^2\left(\boldsymbol{X}_{1|2|3} + \boldsymbol{X}_{2|3|13} + \boldsymbol{X}_{1|3|13} + \boldsymbol{X}_{2|3|12|13} + \boldsymbol{X}_{3|12|13} + \boldsymbol{X}_{3|12|23} + \boldsymbol{X}_{12|23|13}\right)\\
&- Y^2\left(\boldsymbol{X}_{3|12} + \boldsymbol{X}_{1|3|12} + \boldsymbol{X}_{1|2|3|12} + \boldsymbol{X}_{1|2|3|13} + \boldsymbol{X}_{1|3|12|23} + \boldsymbol{X}_{1|12|13|23}\right.\\
&\qquad \left.+\boldsymbol{X}_{2|12|13|23} + \boldsymbol{X}_{3|12|13|23}\right)\\
&+ Y^3\left(-\boldsymbol{X}_{2|3|12|13} + \boldsymbol{X}_{1|2|3|12|13|23}\right).
\end{aligned}$$

◊

Our first main result establishes a general self-reciprocity property for HLS_n.

Theorem A. *We have*

$$\mathsf{HLS}_n(Y^{-1}, \boldsymbol{X}^{-1}) = (-1)^n Y^{-\binom{n}{2}} X_{[n]} \cdot \mathsf{HLS}_n(Y, \boldsymbol{X}).$$

As a corollary we obtain reciprocity results for instantiations of HLS_n. One such result is Corollary 1.11, which establishes a functional equation for Fourier transforms of the Hecke series associated with symplectic groups; see Section 1.4 for details. We prove Theorem A in Section 9.2.

We now present the principal applications of Hall–Littlewood–Schubert series to $\mathfrak{p}$-adic lattice enumeration problems as well as some of their combinatorial and topological properties.

1.2. **Affine Schubert series.** Enumerating full lattices in $\mathbb{Z}^n$ by their index is a classical problem with a well-known solution. The monograph [11] lists no fewer than five proofs of the following identity:

$$\zeta_{\mathbb{Z}^n}(s) := \sum_{\Lambda \leqslant \mathbb{Z}^n} |\mathbb{Z}^n : \Lambda|^{-s} = \prod_{i=0}^{n-1} \zeta(s-i), \tag{1.2}$$

where the sum runs over all lattices of finite index, $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ is the Riemann zeta function and s is a complex variable.

One way to prove (1.2) is to enumerate matrices in Hermite normal form; see [4, Sec. 1]. Its simplicity notwithstanding, this approach has two drawbacks: it is basis-dependent and is oblivious of an important set of intrinsic invariants, namely the *elementary divisors* of Λ with respect to the ambient lattice $\mathbb{Z}^n$.

Enumeration of lattices by their elementary divisors is achieved through suitable specializations of Igusa functions. The ***Igusa function of degree*** n is the rational function in variables $Z_1, \ldots, Z_n$

$$\mathsf{I}_n(Y; Z_1, \ldots, Z_n) = \sum_{I \subseteq [n]} \binom{n}{I}_Y \prod_{\iota \in I} \frac{Z_\iota}{1 - Z_\iota} \in \mathbb{Z}[Y](Z_1, \ldots, Z_n). \tag{1.3}$$

Here, $\binom{n}{I}_Y \in \mathbb{Z}[Y]$ is the Y-multinomial coefficient. The zeta function in (1.2) satisfies the following Euler product decomposition (see [28, Ex. 2.20]):

$$\zeta_{\mathbb{Z}^n}(s) = \prod_{p \text{ prime}} \mathsf{I}_n\left(p^{-1}; \left(p^{i(n-i-s)}\right)_{i \in [n]}\right).$$

Hall–Littlewood–Schubert series may be seen as substantial generalizations of Igusa functions. Indeed, one of their principal applications is to the enumeration of lattices $\Lambda \leqslant \mathbb{Z}^n$ by the elementary divisors of their intersections with all the members of a fixed complete isolated flag of $\mathbb{Z}^n$. As in the case of $\zeta_{\mathbb{Z}^n}(s)$, it suffices to solve this problem locally for all primes p, or equivalently for lattices in $\mathbb{Z}_p^n$, where $\mathbb{Z}_p$ is the ring of p-adic integers. More generally, we consider lattices over a compact discrete valuation ring (cDVR) $\mathfrak{o}$ of arbitrary characteristic.

In this local setup, the relevant elementary divisors are encoded by n partitions, one for each intersection. More precisely, let $V^\bullet = \left(V^{(i)}\right)_{i \in [n]}$ be a complete isolated flag of $\mathfrak{o}^n$; see (2.3). For a lattice $\Lambda \leqslant \mathfrak{o}^n$, denote the type of $\Lambda \cap V^{(i)}$ in $V^{(i)}$ by the partition $\lambda^{(i)}(\Lambda)$ of at most i parts. It determines and is determined by its vector of increments $\mathrm{inc}\left(\lambda^\bullet(\Lambda)\right) = \left(\mathrm{inc}(\lambda^{(i)}(\Lambda)\right)_{i \in [n]} \in \mathbb{N}_0^{\binom{n+1}{2}}$; see (2.1). We introduce $\binom{n+1}{2}$ variables $\boldsymbol{Z} = (Z_{ij})_{1 \leqslant j \leqslant i \leqslant n}$ and set $\boldsymbol{Z}^{\mathrm{inc}(\lambda^\bullet(\Lambda))} = \prod_{i=1}^n \boldsymbol{Z}_i^{\mathrm{inc}(\lambda^{(i)}(\Lambda))}$.

Definition 1.5. The ***affine Schubert series of intersection type*** is

$$\mathrm{affS}_{n,\mathfrak{o}}^{\mathrm{in}}(\boldsymbol{Z}) = \sum_{\Lambda \leqslant \mathfrak{o}^n} \boldsymbol{Z}^{\mathrm{inc}(\lambda^\bullet(\Lambda))} \in \mathbb{Z}[\![\boldsymbol{Z}]\!], \tag{1.4}$$

where the sum runs over all finite-index sublattices Λ of $\mathfrak{o}^n$.

Remark 1.6. The term *affine Schubert series* is a nod to the fact that the defining sum (1.4) may (up to a factor) be interpreted as the generating function of a natural-valued weight function on the vertices of the affine Bruhat–Tits building associated with the group $\mathrm{SL}_n(K)$, where K is the field of fractions of the cDVR $\mathfrak{o}$. Indeed, homothety classes of lattices in K^n form a natural model for the vertex set of the simplicial complex underlying this building. For an early exploitation of this perspective in the enumeration of lattices; see [27]. To what extent affine Schubert series are invariants of affine Schubert varieties remains an interesting open question.

Theorem B shows that the affine Schubert series $\mathrm{affS}^{\mathrm{in}}_{n,\mathfrak{o}}$ is a specialization of the Hall–Littlewood–Schubert series HLS_n. Given $C \subseteq [n]$, we set

$$d_n(C) = \left(\sum_{i\in[n]\setminus C} i\right) - \binom{n - \#C + 1}{2}. \tag{1.5}$$

This is the dimension of the Schubert variety associated with C; see [9, p. 1071]. We denote by $C(k)$ the kth smallest member of C. Set $C(\#C + 1) = n + 1$ and

$$\boldsymbol{Z}_{n,C} = \prod_{k=1}^{\#C} \prod_{\varepsilon=0}^{C(k+1)-C(k)-1} Z_{(C(k)+\varepsilon)k}. \tag{1.6}$$

Note that the (total) degree of $\boldsymbol{Z}_{n,C}$ is $n + 1 - C(1)$.

Theorem B. *For all cDVR $\mathfrak{o}$ with residue field cardinality q we have*

$$\mathrm{affS}^{\mathrm{in}}_{n,\mathfrak{o}}(\boldsymbol{Z}) = \mathsf{HLS}_n(q^{-1}, \left(q^{d_n(C)}\boldsymbol{Z}_{n,C})_C\right).$$

In particular, $\mathrm{affS}^{\mathrm{in}}_{n,\mathfrak{o}}(\boldsymbol{Z})$ is a rational function in $\boldsymbol{Z}$ whose coefficients are polynomials in q. We list these functions for $n \leqslant 3$ in Example A.1. Key to the proof of Theorem B, which we complete in Section 4.3, is to enumerate lattices Λ in $\mathfrak{o}^n$ by associated *intersection tableaux* that, by design, encode the information stored by the partitions $\mathrm{inc}(\lambda^{(i)}(\Lambda))$ for $i \in [n]$.

In Definition 3.15 we define $\mathrm{affS}^{\mathrm{pr}}_{n,\mathfrak{o}}$, a function dual to $\mathrm{affS}^{\mathrm{in}}_{n,\mathfrak{o}}$, recording the elementary divisor types of the projections onto a flag of reference. The duality between the two affine Schubert series is discussed in Section 4.1. By defining a combinatorial operation on pairs of partitions whose skew diagram is a horizontal strip, we show that $\mathrm{affS}^{\mathrm{pr}}_{n,\mathfrak{o}}(\boldsymbol{Z})$ and $\mathrm{affS}^{\mathrm{in}}_{n,\mathfrak{o}}(\boldsymbol{Z})$ are closely related; see also Theorem 4.6. Examples A.1 and A.2 illustrate this proximity for $n \leqslant 3$.

Theorem C. *For all cDVR $\mathfrak{o}$ with residue field cardinality q we have*

$$\mathrm{affS}^{\mathrm{pr}}_{n,\mathfrak{o}}(\boldsymbol{Z}) = \mathsf{HLS}_n\left(q^{-1}, \left(q^{d_n([n]\setminus C)}\boldsymbol{Z}_{n,C}\right)_C\right).$$

Combining Theorem A with Theorems B and C yields that the affine Schubert series also satisfies the following self-reciprocity property:

Corollary 1.7. *We have*

$$\left.\mathrm{affS}^{\mathrm{in}}_{n,\mathfrak{o}}(\boldsymbol{Z}^{-1})\right|_{q\to q^{-1}} = (-1)^n q^{\binom{n}{2}} \left(\prod_{i=1}^{n} Z_{ii}\right) \cdot \mathrm{affS}^{\mathrm{in}}_{n,\mathfrak{o}}(\boldsymbol{Z}),$$

$$\left.\mathrm{affS}^{\mathrm{pr}}_{n,\mathfrak{o}}(\boldsymbol{Z}^{-1})\right|_{q\to q^{-1}} = (-1)^n q^{\binom{n}{2}} \left(\prod_{i=1}^{n} Z_{ii}\right) \cdot \mathrm{affS}^{\mathrm{pr}}_{n,\mathfrak{o}}(\boldsymbol{Z}).$$

1.3. **Hermite–Smith series.** A further substitution of HLS_n pertains to the generating series enumerating lattices in $\mathfrak{o}^n$ according to their elementary divisor types and Hermite composition simultaneously. For the former, let $\lambda(\Lambda)$ be the partition encoding the elementary divisor type of a lattice $\Lambda \leqslant \mathfrak{o}^n$ (see Section 2.3). For the latter, recall that Λ may be represented by a matrix $M \in \mathrm{Mat}_n(\mathfrak{o})$, whose rows record coordinates of generators of Λ with respect to some ordered $\mathfrak{o}$-basis of $\mathfrak{o}^n$. The coset $\mathrm{GL}_n(\mathfrak{o})M$ comprises all such matrices. Let

$$\delta(\Lambda) = (\delta_1(\Lambda), \dots, \delta_n(\Lambda)) \in \mathbb{N}_0^n$$

be the vector of valuations of the diagonal entries of any upper-triangular matrix in $\mathrm{GL}_n(\mathfrak{o})M$. The vector $\delta(\Lambda)$ is in fact an invariant of Λ and the flag $V^\bullet$ whose ith member is generated by the first i elements of the ordered basis. We thus call $\delta(\Lambda)$ the ***Hermite composition*** of Λ relative to $V^\bullet$.

Definition 1.8. For variables $\boldsymbol{x} = (x_1, \dots, x_n)$ and $\boldsymbol{y} = (y_1, \dots, y_n)$ the ***Hermite–Smith series*** is

$$\mathrm{HS}_{n,\mathfrak{o}}(\boldsymbol{x}, \boldsymbol{y}) = \sum_{\Lambda \leqslant \mathfrak{o}^n} \boldsymbol{x}^{\mathrm{inc}(\lambda(\Lambda))} \boldsymbol{y}^{\delta(\Lambda)} \in \mathbb{Z}[\![\boldsymbol{x}, \boldsymbol{y}]\!].$$

Hermite–Smith series are instantiations of Hall–Littlewood–Schubert series:

Theorem D. *For $C \subseteq [n]$, set $C^* = \{n - i + 1 \mid i \in C\}$, and write $\boldsymbol{y}_C = \prod_{i\in C} y_i$. We have*

$$\mathrm{HS}_{n,\mathfrak{o}}(\boldsymbol{x}, \boldsymbol{y}) = \mathsf{HLS}_n\left(q^{-1}, \left(q^{d_n(C)} x_{\#C} \boldsymbol{y}_{C^*}\right)_C\right).$$

In the proof of Theorem D we show that $\mathrm{HS}_{n,\mathfrak{o}}$ factors over $\mathrm{affS}^{\mathrm{in}}_{n,\mathfrak{o}}$; see (4.3).

Theorem B shows that the Igusa function (1.3) is an instantiation of HLS_n.

Corollary 1.9. *We have*

$$\mathsf{I}_n\left(Y, (Z_i)_{i\in[n]}\right) = \mathsf{HLS}_n\left(Y, \left(Y^{d_n([n]\setminus C)} Z_{\#C}\right)\right).$$

In particular, the zeta function $\zeta_{\mathbb{Z}^n,p}(s)$ is equal to

$$\mathsf{I}_n\left(p^{-1}, \left(p^{i(n-i)-is}\right)_{i\in[n]}\right) = \mathrm{HS}_{n,\mathfrak{o}}\left(\left(p^{-is}\right)_{i\in[n]}, \boldsymbol{1}\right) = \mathsf{HLS}_n\left(p^{-1}, (p^{d_n(C)-s\#C})_C\right).$$

Remark 1.10. In Proposition 8.5 we observe that the ***weak order zeta function*** $I_n^{\mathrm{wo}}((X_C)_C)$ (see (8.2)) is also a specialization of HLS_n. Together with the Igusa functions I_n, they are extremal members of the family of *generalized Igusa functions*, introduced and studied in [5]. It would be of great interest to find a framework unifying generalized Igusa functions and Hall–Littlewood–Schubert series.

1.4. **Symplectic Hecke series.** The Hecke series $\tau(Z)$ and its Fourier transforms $\hat{\tau}(\boldsymbol{s}, Z)$ associated with the groups of symplectic similitudes $\mathrm{GSp}_{2n}(F)$ over a local field F are the focus of [12, Sec. V.5], where Z and $\boldsymbol{s} = (s_0, \dots, s_n)$ are variables. In [12, (5.3)] Macdonald gives a formula for $\hat{\tau}(\boldsymbol{s}, Z)$ as a sum of 2^n rational functions in Z and $q^{-s_0}, \dots, q^{-s_n}$, where q is the residue field cardinality of the ring of integers $\mathfrak{o}$ of K. To paraphrase this formula, let $\boldsymbol{x} = (x_0, \dots, x_n)$ and X be variables and set $\boldsymbol{x}_C = \prod_{i\in C} x_i$ for $C \subseteq [n]$. Macdonald exhibits a function

$$\mathsf{H}_{n,\mathfrak{o}}(\boldsymbol{x}, X) = \frac{\mathsf{H}_n^{\mathrm{num}}(q^{-1}, \boldsymbol{x}, X)}{\prod_{I\subseteq[n]}(1 - \boldsymbol{x}_I X)} \in \mathbb{Q}(\boldsymbol{x}, X), \tag{1.7}$$

where $\mathsf{H}_n^{\mathrm{num}}(Y, \boldsymbol{x}, X)$ is a polynomial of degree $2^n - 2$ in X, that satisfies

$$\hat{\tau}(s_0, \dots, s_n, Z) = \mathsf{H}_{n,\mathfrak{o}}(q^{-s_1}, \dots, q^{-s_n}, q^{N-s_0} Z) \tag{1.8}$$

for $N = \frac{1}{4}n(n+1)$. We extend the terminology (*symplectic*) *Hecke series* to the rational functions $\mathsf{H}_{n,\mathfrak{o}}$. We show that they are substitutions of HLS_n.

Theorem E. *For all cDVR $\mathfrak{o}$ with residue field cardinality q we have*

$$\mathsf{H}_{n,\mathfrak{o}}(\boldsymbol{x}, X)(1 - X) = \mathsf{HLS}_n\left(q^{-1}, (\boldsymbol{x}_C X)_C\right).$$

In particular,

$$\mathsf{H}_n^{\mathrm{num}}(Y, \boldsymbol{x}, X) = \sum_{T \in \mathrm{rSSYT}_n} \Phi_T(Y) \prod_{C \in T} \boldsymbol{x}_C X \prod_{\varnothing \neq I \notin T} (1 - \boldsymbol{x}_I X) \in \mathbb{Z}[Y, \boldsymbol{x}, X].$$

We list the numerator polynomials $\mathsf{H}_n^{\mathrm{num}}(Y, \boldsymbol{x}, X)$ for $n \leqslant 3$ in Example A.3. Note that, in addition to providing an alternative to Macdonald's expression, this formula explicates a numerator of the rational function $\hat{\tau}(\boldsymbol{s}, X)$. It also reveals additional properties of the $\mathsf{H}_{n,\mathfrak{o}}$: in Proposition 7.1 we record a simple multiplicative formula for the special value of $\mathsf{H}_{n,\mathfrak{o}}(\boldsymbol{x}, X)$ at $X = 1$. Furthermore, Theorems A and E imply that the Hecke series also satisfies a self-reciprocity property.

Corollary 1.11. *For all cDVR $\mathfrak{o}$ with residue cardinality q we have*

$$\mathsf{H}_{n,\mathfrak{o}}\left(\boldsymbol{x}^{-1}, X^{-1}\right)\big|_{q \to q^{-1}} = (-1)^{n+1} q^{\binom{n}{2}} x_1 \cdots x_n X^2 \cdot \mathsf{H}_{n,\mathfrak{o}}(\boldsymbol{x}, X).$$

Remark 1.12. One can show that the numerator polynomials $\mathsf{N}_n(Y, \boldsymbol{X})$ in (1.1) have no linear term in $\boldsymbol{X}$, that is, the coefficient, as an element of $\mathbb{Z}[Y]$, of X_I is 0 for all non-empty $I \subseteq [n]$. By Theorems A and E, this confirms the conjecture made in [14, Rem. 1.3]. Theorem A confirms the suggestion made in [26, Rem. 4].

1.5. **Quiver representation zeta functions.** A ***quiver*** Q is a finite directed graph with vertex set Q_0 and arrow set Q_1. For $\alpha \in Q_1$, write $h(\alpha) \in Q_0$ and $t(\alpha)$ for the respective head and tail of α: if $\alpha : i \to j$, then $h(\alpha) = i$ and $t(\alpha) = j$. Let R be a commutative ring. An ***R-representation*** of Q is a collection $U = (U_\iota)_{\iota \in Q_0}$ of R-modules U_i, together with an R-module homomorphisms $f_\alpha : U_{t(\alpha)} \to U_{h(\alpha)}$ for each $\alpha \in Q_1$. An R-representation U', with modules U_i' and homomorphisms f_α', is a ***subrepresentation*** of U if $U_j' \leqslant U_j$ with inclusion $\iota_j : U_j' \hookrightarrow U_j$ for all $j \in Q_0$ and $f_\alpha \iota_j = \iota_k f_\alpha'$ for all arrows $\alpha : j \to k$. In this case, we write $U' \leqslant U$. The ***index*** of U' in U is the product of the indices $|U_i : U_i'|$ for each $i \in Q_0$.

The *representation zeta function* $\zeta_U(\boldsymbol{s})$ associated with a fixed R-representation U of a quiver Q was first introduced in [10] for the case when R is a global or local ring of integers and the U_i free, finite-rank R-modules; see [10, (1.1)]. Let $\boldsymbol{s} = (s_i)_{i \in Q_0}$ be complex variables. The representation zeta function is defined as

$$\zeta_U(\boldsymbol{s}) = \sum_{U' \leqslant U} \prod_{i \in Q_0} |U_i : U_i'|^{-s_i}, \tag{1.9}$$

where the sum runs over finite index subrepresentations of U. Certain substitutions of the rational functions HLS_n yield concrete formulae for the (local) representation zeta functions of various quiver representations. We exemplify this with certain representations of dual star quivers.

For $n \in \mathbb{N}$, the *dual star quiver* S_n^* is the quiver with vertex set $[n]$ and arrows $\alpha_i : i \to n$ for all $i \in [n-1]$. See Figure 1.1 for $n = 4$. We define a representation $V_n(\mathfrak{o})$ of S_n^*, as follows: let $V_i = \mathfrak{o}^i$ for all $i \in [n]$, and let $f_{\alpha_i} : \mathfrak{o}^i \to \mathfrak{o}^n$ be an embedding whose images form a complete isolated flag in $V_n = \mathfrak{o}^n$.

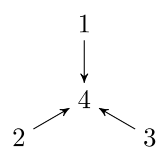


FIGURE 1.1. The dual star quiver S_4^*

Theorem F. *For $C \subseteq [n]$, set $C_0 = C \cup \{0\}$ and let $v_C = (\max(C_0 \cap [i]_0))_{i=1}^n \in \mathbb{N}_0^n$. For the $\mathfrak{o}$-representation $V_n(\mathfrak{o})$ of S_n^* as above, we have*

$$\zeta_{V_n(\mathfrak{o})}(\boldsymbol{s}) = \mathsf{HLS}_n\left(q^{-1}, \left(q^{d_n(C)-v_C\cdot \boldsymbol{s}}\right)_C\right) \prod_{i=1}^{n-1} \zeta_{\mathfrak{o}^i}(s_i).$$

We prove Theorem F in Section 4.5.

1.6. **Tableaux and Bruhat orders.** Hall–Littlewood–Schubert series are defined as finite sums over reduced tableaux. Identifying this index set with the set of chains in a poset opens further combinatorial and topological vantage points.

In Section 6 we define a poset structure $\sqsubseteq$ on the set T_n of non-empty subsets of $[n]$ and explore the topological properties of its associated order complex. In this poset structure, we compare non-empty subsets A and B of $[n]$, written $A \sqsubseteq B$, if A and B arise as labels of adjacent columns in some tableau $T \in \mathrm{SSYT}_n$. This refines the usual containment relation $\supseteq$ on $[n]$: if $A \supseteq B$, then $A \sqsubseteq B$. The order complex $\Delta(\mathsf{T}_n)$ associated with the poset T_n is, as simplicial complex, isomorphic to the set of rSSYT_n of reduced tableaux with labels in $[n]$. We denote by $|\Delta(\mathsf{T}_n)|$ a geometric realization of $\Delta(\mathsf{T}_n)$.

Theorem 1.13. *The simplicial complex $|\Delta(\mathsf{T}_n)|$ is Cohen–Macaulay over $\mathbb{Z}$ and homeomorphic to an $\left(\binom{n+1}{2} - 1\right)$-ball. The number of maximal flags in $\Delta(\mathsf{T}_n)$ is*

$$\frac{\binom{n+1}{2}! \cdot \prod_{a=1}^{n-1}(a!)}{\prod_{b=1}^{n}((2b-1)!)}.$$

We prove Theorem 1.13 in Section 6.3.1. At the heart of the proof is a poset isomorphism between T_n and a poset arising from the parabolic quotient of the hyperoctahedral group of degree n by its maximal symmetric group. The partial order on that set is given by the Bruhat order.

1.7. **Main ideas and structure of the paper.** Section 2 sets up some essential notation. It contains a table of notation and may serve as a reference throughout.

Section 3 is central to the paper's methodology. We associate with a full lattice Λ in $V \cong \mathfrak{o}^n$ two types of tableaux, viz. the intersection tableaux $T^{\bullet}(\Lambda)$ and the projection tableaux $T_{\bullet}(\Lambda)$, both in SSYT_n. These tableaux encode how the lattice Λ is built up successively from the intersections and the projections relative to the members of a flag in V by cyclic extensions.

In Theorem 4.6 we enumerate the fibers of these two maps in terms of two combinatorial invariants of a tableau $T \in \mathrm{SSYT}_n$. The first is the leg polynomial $\Phi_T(Y)$ introduced in Definition 1.1, and the second is the sum $D_n(T)$ of the dimensions of the Schubert varieties indexed by the columns of T; see Definition 4.2. Since Hall–Littlewood–Schubert series are defined as sums over tableaux that record their column structure weighted by their leg polynomials, this paves the way to the proofs of Theorems B and C in Section 4.3 and of Theorem D in Section 4.4. The proof of Theorem F is given in Section 4.5.

In Section 5 we give an interpretation of the leg polynomials $\Phi_T(Y)$ in terms of Dyck words. It is logically independent from any of the paper's main theorems and may be of independent combinatorial interest.

Hall–Littlewood–Schubert series are Y-analogs of Stanley–Reisner rings of a simplicial complex, namely the order complex $\Delta(\mathsf{T}_n)$ of the poset T_n we define in Section 6.1. The observation that $\Delta(\mathsf{T}_n)$ is isomorphic to the poset rSSYT_n, the finite indexing set of the sum defining HLS_n, is crucial for the rest of this section; see Lemma 6.1. In Proposition 6.7 we establish an isomorphism between $\mathsf{T}_n \cup \{\varnothing\}$ and a parabolic quotient of the hyperoctahedral group B_n of degree n. This is an important waypoint towards the proof of Theorem 1.13 in Section 6.3.1. En route we leverage the well-known bijection between tableaux and Gelfand–Tsetlin patterns to obtain quantitative statements about maximal reduced tableaux.

In Section 7 we develop the connections between Hall–Littlewood–Schubert series and symplectic Hecke series. Apart from our proof of Theorem E, we record in Section 7.2 a simple multiplicative formula for the special value of the Hecke series at $X = 1$ (see Proposition 7.1), generalizing Schur's classical Littlewood identities.

Special values of Hall–Littlewood–Schubert series are the theme of Section 8. We focus on univariate series obtained by setting all the X_C to X and Y to one of 0, 1, or -1. In the case $Y = 0$ the Cohen–Macaulay property of certain (Stanley–Reisner) rings implies the non-negativity of the relevant series' numerators; see Proposition 8.2. In the other cases $Y = 1$ or $Y = -1$, we formulate non-negativity conjectures that seem to transcend the remit of Stanley–Reisner rings of simplicial complexes; see Conjectures 8.3 and 8.7.

In Section 9 we explore different interpretations of Hall–Littlewood–Schubert series as $\mathfrak{p}$-adic integrals. In Section 9.1 we show that classical integrals over the integral $\mathfrak{p}$-adic points of groups of symplectic similitudes are instances of Hall–Littlewood–Schubert series. This yields a simplified proof of a combinatorial identity for these integrals in terms of Igusa functions, previously proven in [1]. We use a different expression of HLS_n as a $\mathfrak{p}$-adic integral to prove Theorem A in Section 9.3.

2. Notation

We recall some standard definitions and notation from the theories of integer partitions, tableaux, and lattices. We write $\mathbb{N} = \{1, 2, \dots\}$, and for $I \subseteq \mathbb{N}$, set $I_0 = I \cup \{0\}$. For $a, b \in \mathbb{N}$, let $[a, b] = \{a, a + 1, \dots, b\}$ and $[a] = [1, a]$. For $C \subseteq [n]$ and $k \in \mathbb{N}$, let $C(k)$ be the kth smallest member of C. We also set $C(\#C+1) = n+1$. For a variable Y, we set $\binom{n}{\varnothing}_Y = 1$, and for $I \subseteq [n]$ with $k = \max(I)$, set

$$\binom{n}{I}_Y = \binom{n}{k}_Y \binom{k}{I\backslash\{k\}}_Y = \frac{(1-Y^n)(1-Y^{n-1})\cdots(1-Y^{n-k+1})}{(1-Y)(1-Y^2)\cdots(1-Y^k)} \cdot \binom{k}{I\backslash\{k\}}_Y.$$

For $I \subseteq [n]$, set $\operatorname{maj}(I) = \sum_{i\in I} i$. For $u = (u_1, \dots, u_m) \in \mathbb{Z}^m$, set $|u| = \sum_{i=1}^m u_i \in \mathbb{Z}$.

2.1. Partitions.

A ***partition*** λ is a weakly decreasing sequence $(\lambda_i)_{i\in\mathbb{N}}$ such that each $\lambda_i \in \mathbb{N}_0$ and all but finitely many λ_i are zero. The ***parts*** of λ are the positive λ_i for $i \in [n]$. We denote by $\mathcal{P}_n$ the set of all partitions with at most n parts. We sometimes write $\lambda = (\lambda_1, \dots, \lambda_n)$ to mean $\lambda = (\lambda_1, \dots, \lambda_n, 0, \dots)$.

A partition $\lambda \in \mathcal{P}_n$ determines and is determined by its ***Young diagram***, which is a collection of $|\lambda|$ cells, arranged in at most n left-justified rows with λ_i cells in the ith row, starting at the top and going down. The ***shape*** of a Young diagram with n rows is the partition λ such that λ_i is the number of cells in row i. The ***conjugate*** partition λ' corresponds to the Young diagram associated with λ, reflected along the main diagonal: λ'_i is the number of cells in the ith column.

Given $\lambda, \mu \in \mathcal{P}_n$, we write $\mu \subseteq \lambda$ if $\mu_i \leqslant \lambda_i$ for each $i \in [n]$. In this case, we may superimpose the two Young diagrams by aligning the top left corners, with the Young diagram of shape μ inside of the Young diagram of shape λ. The ***skew diagram*** $\lambda - \mu$ is the diagram obtained from the Young diagram with shape λ by removing all of the cells in μ. A skew diagram is a ***horizontal strip*** if there is at most one cell in each of its columns. We write HoSt_n for the set of pairs of partitions $(\lambda, \mu) \in \mathcal{P}_n \times \mathcal{P}_{n-1}$ such that $\mu \subseteq \lambda$ and $\lambda - \mu$ is a horizontal strip.

For $\lambda \in \mathcal{P}_n$, set

$$\operatorname{inc}(\lambda) = (\lambda_1 - \lambda_2,\ \dots,\ \lambda_{n-1} - \lambda_n,\ \lambda_n) = (\operatorname{inc}_i(\lambda))_{i\in[n]} \in \mathbb{N}_0^n. \tag{2.1}$$

Given a sequence of partitions $\lambda^\bullet = (\lambda^{(1)}, \dots, \lambda^{(n)})$, we write

$$\mathrm{inc}(\lambda^\bullet) = (\mathrm{inc}(\lambda^{(1)}),\ \dots,\ \mathrm{inc}(\lambda^{(n)})).$$

2.2. **Tableaux.** A (***semistandard Young***) ***tableau*** T is a Young diagram where each cell is filled in with a natural number such that the values are non-decreasing across each row and increasing down each column. For $i, j \in \mathbb{N}$, the (i,j)-***cell*** of T is—if it exists—the cell in the ith row and jth column; its entry is written T_{ij}. The ***shape*** of T is that of its underlying Young diagram, written $\mathrm{sh}(T)$. We write $T = (C_1, C_2, \dots)$ for subsets $C_j \subseteq [n]$ to denote the columns of T, where

$$C_j = \{T_{i,j} \mid i \in [n]\} = \{C_j(k) \mid k \in [\#C_j]\} \subseteq [n].$$

We write $C \in T$ to express that C is a column of T. The ***weight*** of T is the vector $\mathrm{wt}(T) = (\omega_1, \dots, \omega_n) \in \mathbb{N}_0^n$ if T has exactly ω_i cells with entry i. A tableau is ***reduced*** if it contains no repeated columns. We write SSYT_n for the set of tableaux with entries in $[n]$ and $\mathrm{rSSYT}_n \subseteq \mathrm{SSYT}_n$ for the set of reduced tableaux.

Tableaux are in bijection with flags of partitions where all the skew diagrams associated with successive pairs of partitions are horizontal strips. For $T \in \mathrm{SSYT}_n$ and $k \in [n]_0$, let $T^{(k)}$ be the tableau obtained from T by removing all cells with labels greater than k. The shape of $T^{(k)}$ is the partition $\lambda^{(k)}(T) = (\lambda_1^{(k)}, \dots, \lambda_k^{(k)}) \in \mathcal{P}_k$. This yields a flag of partitions, namely

$$\lambda^\bullet(T): \quad () = \lambda^{(0)}(T) \subseteq \lambda^{(1)}(T) \subseteq \dots \subseteq \lambda^{(n-1)}(T) \subseteq \lambda^{(n)}(T) = \lambda. \tag{2.2}$$

By the tableau condition, differences of successive pairs of these partitions are horizontal strips. Conversely, given a flag of partitions

$$() = \lambda^{(0)} \subseteq \lambda^{(1)} \subseteq \dots \subseteq \lambda^{(n-1)} \subseteq \lambda^{(n)},$$

where each successive skew diagram $\lambda^{(i)} - \lambda^{(i-1)}$ is a horizontal strip, we obtain a tableau T of shape $\lambda^{(n)}$ by labelling cells in the ith horizontal strip by i, for each $i \in [n]$. These two constructions are mutually inverse.

2.3. **Lattices and isolated (co-)flags.** Let $\mathfrak{o}$ be a cDVR of arbitrary characteristic and K its field of fractions. Let $\mathfrak{p} \subset \mathfrak{o}$ be the unique maximal ideal and $\pi \in \mathfrak{p}$ a uniformizing element. We assume that the residue field $\mathfrak{o}/\mathfrak{p}$ has characteristic p and cardinality q, sometimes written $\mathbb{F}_q$.

Fix throughout a free $\mathfrak{o}$-module V of finite rank n. An $\mathfrak{o}$-***lattice*** $\Lambda \leqslant V$ is a $\mathfrak{o}$-submodule of V of full rank n. We write $\mathcal{L}(V)$ for the set of all such lattices. For each $i \in [n-1]_0$, let U_i be a free $\mathfrak{o}$-module. A ***complete isolated flag*** is

$$U_0 \xhookrightarrow{\iota_1} U_1 \xhookrightarrow{\iota_2} \cdots \xhookrightarrow{\iota_n} U_n,$$

where each U_i is an $\mathfrak{o}$-module with rank i and the cokernel of each ι_j is torsion-free. Dually, a ***complete isolated coflag*** is

$$U_0 \xtwoheadleftarrow{\varpi_0} U_1 \xtwoheadleftarrow{\varpi_1} \cdots \xtwoheadleftarrow{\varpi_{n-1}} U_n,$$

where each U_i is an $\mathfrak{o}$-module with rank i and the coimage of each ϖ_j is torsion-free. Fix once and for all a complete isolated flag and a complete isolated coflag of V:

$$\begin{aligned} V^\bullet: &\quad 0 = V^{(0)} \xhookrightarrow{\iota_1} V^{(1)} \xhookrightarrow{\iota_2} \cdots \xhookrightarrow{\iota_n} V^{(n)} = V, \\ V_\bullet: &\quad 0 = V_{(0)} \xtwoheadleftarrow{\varpi_0} V_{(1)} \xtwoheadleftarrow{\varpi_1} \cdots \xtwoheadleftarrow{\varpi_{n-1}} V_{(n)} = V. \end{aligned} \tag{2.3}$$

We assume, without loss of generality, that each $V^{(i)}$ is a submodule of V and each $V_{(i)}$ is a quotient of V.

For a partition λ, we define the finite $\mathfrak{o}$-module $C_\lambda(\mathfrak{o}) = \bigoplus_{i\in\mathbb{N}} \mathfrak{o}/\mathfrak{p}^{\lambda_i}$. The (***elementary divisor***) ***type*** of a lattice $\Lambda \in \mathcal{L}(V)$, written $\lambda(\Lambda)$, is the partition

λ with $V/\Lambda \cong C_\lambda(\mathfrak{o})$. Let $w \in C_\lambda(\mathfrak{o})$ and $k \in \mathbb{N}_0$. Write $v_\mathfrak{p}(w) \geqslant k$ if $w \in \mathfrak{p}^k C_\lambda(\mathfrak{o})$. For an $\mathfrak{o}$-module M, define

$$\mathrm{Ann}_M(\mathfrak{p}^k) = \left\{ m \in M \mid \mathfrak{p}^k m = 0 \right\}.$$

If $M = C_\lambda(\mathfrak{o})$, then

$$\lambda_i' = \dim_{\mathbb{F}_q} \left(\mathfrak{p}^{i-1}M/\mathfrak{p}^i M \right) = \dim_{\mathbb{F}_q} \left(\mathrm{Ann}_M(\mathfrak{p}^i)/\mathrm{Ann}_M(\mathfrak{p}^{i-1}) \right).$$

The proof for the following lemma is essentially due to [12, II.4 (4.12)].

Lemma 2.1. *Let $\lambda, \mu \in \mathcal{P}_n$. Let M be a finite $\mathfrak{o}$-module of type λ and $N \leqslant M$ of type μ. If M/N is cyclic, then $\mu \subseteq \lambda$ and $\lambda - \mu$ is a horizontal strip.*

Proof. Since N is a submodule, it follows that $\mu \subseteq \lambda$. Fix $k \in \mathbb{N}$. Note that $\mathrm{Ann}_N(\mathfrak{p}^k) = N \cap \mathrm{Ann}_M(\mathfrak{p}^k)$ and $\mathfrak{p}\mathrm{Ann}_M(\mathfrak{p}^k) \subseteq \mathrm{Ann}_M(\mathfrak{p}^{k-1})$. Since $\mathrm{Ann}_N(\mathfrak{p}^{k-1}) = \mathrm{Ann}_N(\mathfrak{p}^k) \cap \mathrm{Ann}_M(\mathfrak{p}^{k-1})$, we have

$$\dim_{\mathbb{F}_q} \left((\mathrm{Ann}_N(\mathfrak{p}^k) + \mathrm{Ann}_M(\mathfrak{p}^{k-1}))/\mathrm{Ann}_M(\mathfrak{p}^{k-1}) \right) = \mu_k'.$$

As $\dim_{\mathbb{F}_q} \left(\mathrm{Ann}_M(\mathfrak{p}^k)/\mathrm{Ann}_M(\mathfrak{p}^{k-1}) \right) = \lambda_k'$, we have

$$\dim_{\mathbb{F}_q} \left(\mathrm{Ann}_M(\mathfrak{p}^k)/(\mathrm{Ann}_N(\mathfrak{p}^k) + \mathrm{Ann}_M(\mathfrak{p}^{k-1})) \right) = \lambda_k' - \mu_k'. \tag{2.4}$$

Since $\mathrm{Ann}_M(\mathfrak{p}^k)/\mathrm{Ann}_N(\mathfrak{p}^k) \cong (N + \mathrm{Ann}_M(\mathfrak{p}^k))/N$ and since M/N is cyclic, the $\mathbb{F}_q$-vector space in (2.4) is also cyclic. Hence, $\lambda_k' - \mu_k' \leqslant 1$ for all $k \in \mathbb{N}$. □

2.4. **Further notation.** We record further notation in the following table.

Symbol	Description	Reference
$\mathcal{P}_n$	partitions with at most n parts	Section 2.1
HoSt_n	partitions $\mu \subseteq \lambda$ yielding horizontal strips	Section 2.1
$\mathcal{L}(V)$	set of full lattices in V	Section 2.3
$\delta(\Lambda)$	Hermite composition	Section 1.3
SSYT_n	tableaux with labels in $[n]$	Section 2.2
rSSYT_n	tableaux without repeated columns	Section 2.2
$\Phi_T(Y)$	Leg polynomial	Definition 1.1
$\mathsf{HLS}_n(Y, \boldsymbol{x})$	Hall–Littlewood–Schubert series	Definition 1.2
$\mathrm{affS}^{\mathrm{in}}_{n,\mathfrak{o}}(\boldsymbol{Z})$	affine Schubert series of intersection type	Definition 1.5
$\mathrm{affS}^{\mathrm{pr}}_{n,\mathfrak{o}}(\boldsymbol{Z})$	affine Schubert series of projection type	Definition 3.15
$\mathsf{H}_{n,\mathfrak{o}}(\boldsymbol{x}, X)$	Fourier transform of Hecke series	(1.7)
$\lambda^\bullet(T)$	flag of partitions of tableau T	(2.2)
$\lambda^\bullet(\Lambda)$	flag of partitions of intersection types for Λ	(3.7)
$T^\bullet(\Lambda)$	intersection tableau for Λ	Section 3.3
$\lambda_\bullet(\Lambda)$	flag of partitions of projection types for Λ	(3.13)
$T_\bullet(\Lambda)$	projection tableau for Λ	Section 3.5
$V^\bullet$	complete isolated flag of submodules of V	(2.3)
$V_\bullet$	complete isolated coflag of quotients of V	(2.3)
$\mathcal{L}^{\mathrm{in}}_T(V)$	lattices with intersection data given by T	(3.8)
$\mathcal{L}^{\mathrm{pr}}_T(V)$	lattices with projection data given by T	(3.14)
$\mathcal{D}$	finite Dyck words	Section 5
$d_n(C)$	dimension of Schubert variety for $C \subseteq [n]$	(1.5)
$D_k(T)$, $D_k^{\mathrm{c}}(T)$	(dual) kth Schubert dimension	Definition 4.2
T_n	poset on $2^{[n]} \backslash \{\varnothing\}$ with tableau order	Section 6.1

$\Delta(\mathsf{T}_n)$	order complex of T_n	Section 6.1
SR_n	Stanley–Reisner ring associated with T_n	Section 8.1
B_n	hyperoctahedral group of degree n	Section 6.2
GT_n	Gelfand–Tsetlin patterns of degree n	Section 6.3

3. Cyclic extensions of lattices

In this section we prepare the groundwork for our application of Hall–Littlewood–Schubert series to affine Schubert series of intersection and projection type. In both cases, we construct a surjective map from the set $\mathcal{L}(V)$ of finite-index lattices of V to the set SSYT_n of tableaux comprising the information recorded in the affine Schubert series of the respective type; see Sections 3.3 and 3.5. In Section 4 we provide formulae for the cardinalities of the fibers of these maps. Sections 3.1 and 3.2 contain preliminary notation and results on partitions and lattice extensions.

3.1. Corners, gaps, and jigsaws.

Before we enumerate lattices by either intersection or projection data, we establish a few combinatorial results concerning pairs of partitions. The section's examples and figures illustrate them.

The (i,j)-cell of the Young diagram of shape $\lambda \in \mathcal{P}_n$ is a ***corner*** if there is no $(i+1,j)$-cell and no $(i,j+1)$-cell in the diagram. Corners have the form (λ'_a, a) for some $a \in \mathbb{N}$ and, equivalently, (b, λ_b) for some $b \in \mathbb{N}$.

Let σ be a horizontal strip and $(\lambda,\mu) \in \mathrm{HoSt}_n$; see Section 2.3. We write

$$C_{\lambda,\mu} = \left\{(\mu'_a, a) \;\middle|\; a \in \mathbb{N},\ \mu'_a = \lambda'_a,\ \mu'_{a+1} < \lambda'_{a+1}\right\} \subseteq [n-1] \times [\mu_1].$$

for the set of the corners of μ within columns not containing a cell of σ and immediately preceding a column containing a cell from σ. The cells in $C_{\lambda,\mu}$ are, in other words, those immediately to the west of maximal contiguous substrips of σ. Write π_1 for the projection onto the first coordinate of elements in $C_{\lambda,\mu}$ and π_2 for the projection onto the second coordinate. Lets us define the sets

$$I_{\lambda,\mu} = \pi_1(C_{\lambda,\mu}), \quad J_{\lambda,\mu} = \pi_2(C_{\lambda,\mu}).$$

Since there is at most one corner in each row and column, $\#C_{\lambda,\mu} = \#I_{\lambda,\mu} = \#J_{\lambda,\mu}$.

We define two additional partitions $\nu,\gamma \in \mathcal{P}_{n-1}$, whose parts are given by

$$\nu_i = \sum_{j>i}(\lambda_j - \mu_j), \qquad \gamma_i = \mu_i - \nu_i = \mu_i - \sum_{j>i}(\lambda_j - \mu_j).$$

We note that ν_i is the number of cells of λ in row i lying above (but not on) a cell of the horizontal strip σ, whereas γ_i is the number of such cells lying neither on nor above a cell of σ. We call ν the ***valuation partition*** and γ the ***gap partition*** associated with (λ,μ). The number of cells in λ lying neither on nor above a cell of σ is denoted

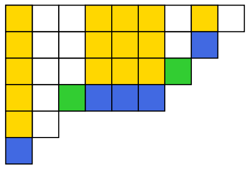
$$\mathrm{gap}(\lambda,\mu) = |\gamma|. \tag{3.1}$$

Hence, $\mathrm{gap}(\lambda,\mu)$ counts the number of cells in columns of λ in the gaps between the contiguous strips of σ.

Example 3.1. For $n = 6$ and $\lambda = (9,8,7,6,2,1)$ and $\mu = (9,7,7,3,2)$, we illustrate the valuation and gap partitions in the Young diagram for λ in Figure 3.1. We color the cells of $\sigma := \lambda - \mu$ in blue, the cells of $C_{\lambda,\mu} = \{(3,7),(4,3)\}$ in green, and the cells above a cell of σ in yellow. The number of yellow cells in row i is γ_i, and the number of white or green cells in row i is ν_i. We note that $\mathrm{gap}(\lambda,\mu) = 13$. ♢

Lemma 3.2. *Let $(\lambda,\mu) \in \mathrm{HoSt}_n$ with associated valuation and gap partitions ν,γ, and set $m = \max(I_{\lambda,\mu})$. Suppose*

$$a = \min\left(\{k \in \mathbb{N} \mid \nu_k < |\sigma|\} \cup \{\infty\}\right), \qquad b = \min\{k \in \mathbb{N} \mid \gamma_k = 0\}.$$



$\lambda = (9,8,7,6,2,1) \in \mathcal{P}_6,$
$\mu = (9,7,7,3,2) \in \mathcal{P}_5,$
$\sigma = (0,1,0,3,0,1) \in \mathbb{N}_0^6,$
$\nu = (5,4,4,1,1) \in \mathcal{P}_5,$
$\gamma = (4,3,3,2,1) \in \mathcal{P}_5.$

FIGURE 3.1. Valuation and gap partitions for $(\lambda, \mu) \in \mathrm{HoSt}_6$

Then $a \geqslant b$ if and only if $C_{\lambda,\mu} = \varnothing$. If $a < b$, then $\nu_{b-1} = \nu_m$.

Proof. The case when $a = \infty$ is clear, so assume $b \leqslant a < \infty$. Then $\nu_{a-1} = |\sigma|$ if and only if $\lambda_k = \mu_k$ for all $k \in [a-1]$. Hence, $[a-1] \cap I_{\lambda,\mu} = \varnothing$. Since $\gamma_a = 0$ and $\nu_{a-1} = |\sigma|$, we have $\lambda_a = |\sigma|$. Thus, $[\lambda_a] \cap J_{\lambda,\mu} = \varnothing$. Therefore, $C_{\lambda,\mu} = \varnothing$.

Conversely, suppose $C_{\lambda,\mu} = \varnothing$. If $|\sigma| = 0$, then we are done, so suppose $|\sigma| \in \mathbb{N}$. Then there exists $k \in \mathbb{N}$ such that $\lambda_i' \neq \mu_i'$ for all $i \in [k]$ and $\lambda_j' = \mu_j'$ for all $j > k$. In other words, there exists $r \in \mathbb{N}$ such that $\lambda_i = \mu_i$ for all $i \in [r-1]$ and $\lambda_r = |\sigma|$. Thus $\nu_{r-1} = |\sigma|$, so $\mu_r - \nu_r = \mu_r - (|\sigma| - \lambda_r + \mu_r) = 0$. Hence, $\gamma_r = 0$ and thus $a \geqslant b$.

For the final claim, suppose $a < b$. Since $\gamma_b = 0$ implies $\mu_b = \nu_b$, we have $m \leqslant b-1$, so assume $m < b-1$. Since $\gamma_{b-1} > 0$, we have $0 < \gamma_{b-1} - \gamma_b = \mu_{b-1} - \lambda_b$. Hence, $\mu_{b-1} > \lambda_b \geqslant \mu_b$, so that $(b-1, \mu_{b-1})$ is a corner of μ. For $r = \mu_{b-1}$, we have $\lambda_r' = \mu_r'$. For $s = \mu_m$, we also have $\lambda_s' = \mu_s'$. By maximality of m, we have $\lambda_i' = \mu_i'$ for all $i \in [r, s]$. Since (m, μ_m) and $(b-1, \mu_b)$ are corners, this implies $\lambda_j = \mu_j$ for all $j \in [m+1, b-1]$. Hence,

$$0 = \sum_{j=m+1}^{b-1} (\lambda_j - \mu_j) = \nu_m - \nu_{b-1}. \qquad \square$$

Given $(\lambda, \mu) \in \mathrm{HoSt}_n$, we define $(\widetilde{\lambda}, \widetilde{\mu}) \in \mathcal{P}_n \times \mathcal{P}_{n-1}$ by

$$\begin{aligned} \widetilde{\lambda} &= \lambda_1 - \lambda = (\lambda_1 - \lambda_n, \lambda_1 - \lambda_{n-1}, \dots, \lambda_1 - \lambda_1), \\ \widetilde{\mu} &= \lambda_1 - \mu = (\lambda_1 - \mu_{n-1}, \lambda_1 - \mu_{n-2}, \dots, \lambda_1 - \mu_1). \end{aligned} \tag{3.2}$$

Let $\rho \in \mathcal{P}_n$ with $\rho_1 = \rho_n = \lambda_1$. The *jigsaw operation* in (3.2) for λ can be visualized by reflecting the skew diagram $\rho - \lambda$ vertically and horizontally and for μ with the same reflections applied to the skew diagram $\rho - (\lambda_1, \mu_1, \dots, \mu_{n-1})$. For all $i \in [\mu_1]$,

$$\lambda_i' + \widetilde{\lambda}_{n-i+1}' = n, \qquad \mu_i' + \widetilde{\mu}_{n-i+1}' = n-1. \tag{3.3}$$

This implies that $\widetilde{\lambda} - \widetilde{\mu}$ is a horizontal strip, so $(\widetilde{\lambda}, \widetilde{\mu}) \in \mathrm{HoSt}_n$. Moreover,

$$\lambda_i' - \mu_i' = 0 \qquad \Longleftrightarrow \qquad \widetilde{\lambda}_{n-i+1}' - \widetilde{\mu}_{n-i+1}' = 1. \tag{3.4}$$

Hence $\mathrm{gap}(\widetilde{\lambda}, \widetilde{\mu})$ is the number of cells in ρ below a cell of σ, or equivalently

$$\mathrm{gap}(\widetilde{\lambda}, \widetilde{\mu}) = n|\sigma| - |\nu|, \tag{3.5}$$

where ν is the valuation partition associated with λ and μ.

Lemma 3.3. *The map $J_{\lambda,\mu} \to J_{\widetilde{\lambda},\widetilde{\mu}}$ given by $a \mapsto n-a$ is a bijection, and*

$$\mathrm{inc}_a(\mu') = \mathrm{inc}_{n-a}(\widetilde{\mu}').$$

Proof. Suppose $(\mu_a', a) \in C_{\lambda,\mu}$, so $\lambda_a' = \mu_a'$ and $\lambda_{a+1}' \neq \mu_{a+1}'$. By (3.4),

$$\widetilde{\lambda}_{n-a+1}' \neq \widetilde{\mu}_{n-a+1}', \qquad \widetilde{\lambda}_{n-a}' = \widetilde{\mu}_{n-a}'.$$

So $(\widetilde{\mu}'_{n-a}, n-a) \in C_{\widetilde{\lambda},\widetilde{\mu}}$. Thus, $C_{\lambda,\mu}$ and $C_{\widetilde{\lambda},\widetilde{\mu}}$ are in bijection. By (3.3),

$$\mathrm{inc}_a(\mu') = \mu'_a - \mu'_{a+1} = \widetilde{\mu}'_{n-a} - \widetilde{\mu}'_{n-a+1} = \mathrm{inc}_{n-a}(\widetilde{\mu}'). \qquad \square$$

Example 3.4. We illustrate the jigsaw operation defined in (3.2) with the partitions from Example 3.1: $\lambda = (9,8,7,6,2,1)$ and $\mu = (9,7,7,3,2)$ yield $\widetilde{\lambda} = (8,7,3,2,1,0)$ and $\widetilde{\mu} = (7,6,2,2,0)$. Figure 3.2 shows the pairs $(\lambda,\mu), (\widetilde{\lambda},\widetilde{\mu}) \in \mathrm{HoSt}_6$. There we color the cells of $\lambda - \mu$ in blue, $C_{\lambda,\mu}$ in green, $\widetilde{\lambda} - \widetilde{\mu}$ in purple, and $C_{\widetilde{\lambda},\widetilde{\mu}}$ in orange. $\diamondsuit$

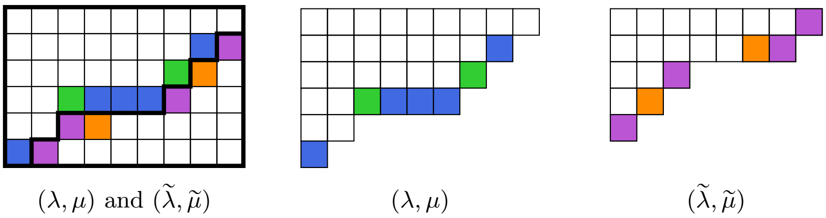


FIGURE 3.2. An illustration of the jigsaw operation (3.2)

3.2. **Extending elements for lattices.** In this section we study ways to extend lattices of types λ and to lattices of type μ by adjoining suitable elements. For a partition λ, we set $D_\lambda = \{i \in \mathbb{N} \mid \lambda_i > \lambda_{i+1}\} = \{\lambda'_j \mid j \in \mathbb{N}\}$. For $i \in D_\lambda$, we set $e_i = \mathrm{inc}_{\lambda_i}(\lambda') \in \mathbb{N}$. Then

$$C_\lambda(\mathfrak{o}) = \bigoplus_{i \in D_\lambda} C_{\lambda_i}(\mathfrak{o})^{e_i}.$$

For each $i \in D_\lambda$ we let define the projection

$$\vartheta_{\lambda,i} : C_\lambda(\mathfrak{o}) \to C_{\lambda_i}(\mathfrak{o})^{e_i}, \quad (w_1, w_2, \dots) \mapsto (w_{i-e_i+1}, w_{i-e_i+2}, \dots, w_i). \tag{3.6}$$

Given a partition β, we generalize the projections $\vartheta_{\lambda,i}$ to projections

$$\vartheta^\beta_{\lambda,i} : C_\beta(\mathfrak{o}) \to \bigoplus_{j=1}^{e_i} C_{\beta_{i-j+1}}(\mathfrak{o})$$

given by the same formula in (3.6). Thus $\vartheta^\lambda_{\lambda,i} = \vartheta_{\lambda,i}$. We have two specific use cases for β: for intersection data, $\beta = \mu$, and for projection data, $\beta = (|\lambda - \mu|^{(n-1)})$.

Definition 3.5. Let $(\lambda,\mu) \in \mathrm{HoSt}_n$ with valuation partition $\nu \in \mathcal{P}_{n-1}$. An element $w \in C_\beta(\mathfrak{o})$ is (λ,μ)-***extending*** if for all $i \in D_\mu$ and all $j \in I_{\lambda,\mu}$,

$$v_\mathfrak{p}\left(\vartheta^\beta_{\mu,i}(w)\right) \geqslant \nu_i, \qquad\qquad v_\mathfrak{p}\left(\vartheta^\beta_{\mu,j}(w)\right) \leqslant \nu_j.$$

We call the inequalities in Definition 3.5 "Condition (1)" and "Condition (2)".

Lemma 3.6. *Let $(\lambda,\mu) \in \mathrm{HoSt}_n$ with $\mu_{n-1} \neq 0$. Let $\beta \in \mathcal{P}_{n-1}$ with $\beta_{n-1} \geqslant \lambda_n$. If $w \in C_\beta(\mathfrak{o})$ is (λ,μ)-extending, then $v_\mathfrak{p}(w) \geqslant \lambda_n$ and either $\lambda_n = \mu_{n-1}$, $\mu = (\lambda_1, \dots, \lambda_{n-1})$, or $v_\mathfrak{p}(w) = \lambda_n$.*

Proof. Let $\nu, \gamma \in \mathcal{P}_{n-1}$ be the valuation and gap partitions associated with (λ,μ). As $\mu_{n-1} \neq 0$, we have $n-1 \in D_\mu$. Since $\min\{\nu_1, \dots, \nu_{n-1}\} = \nu_{n-1} = \lambda_n$, by Condition (1) of Definition 3.5 we have that $v_\mathfrak{p}(w) \geqslant \lambda_n$.

Assume that $\lambda_n \neq \mu_{n-1}$ and $\mu \neq (\lambda_1, \dots, \lambda_{n-1})$, so we will show that $v_\mathfrak{p}(w) = \lambda_n$. Note that $\lambda_n \neq \mu_{n-1}$ implies that

$$\gamma_{n-1} = \mu_{n-1} - \lambda_n > 0,$$

and $\mu \neq (\lambda_1, \dots, \lambda_{n-1})$ implies that

$$\nu_{n-1} = \lambda_n < |\lambda| - |\mu|.$$

By Lemma 3.2 with $a \leqslant n-1$ and $b = n$, we have $I_{\lambda,\mu} \neq \varnothing$. Moreover for $m = \max(I_{\lambda,\mu})$, Lemma 3.2 also implies that $\nu_m = \nu_{n-1} = \lambda_n$. Since $\beta_n \geqslant \lambda_n$, we have $v_{\mathfrak{p}}(\vartheta_{\mu,n-1}(w)) = \lambda_n$ by Condition (2) of Definition 3.5. Hence, $v_{\mathfrak{p}}(w) = \lambda_n$. □

Lemma 3.7. *Let $(\lambda, \mu) \in \mathrm{HoSt}_n$. Let $\rho = (\lambda_1, \dots, \lambda_{n-1})$ and $\tau = (\mu_1, \dots, \mu_{n-2})$. Let $\beta \in \mathcal{P}_{n-1}$ and $\alpha = (\beta_1, \dots, \beta_{n-2})$. With $\varpi : C_\beta(\mathfrak{o}) \to C_\alpha(\mathfrak{o})$ given by $(w_1, \dots, w_{n-1}) \mapsto (w_1, \dots, w_{n-2})$, if $w \in C_\beta(\mathfrak{o})$ is (λ, μ)-extending, then $\varpi(w)$ is (ρ, τ)-extending.*

Proof. If either $\mu_{n-1} = 0$ or $\mu_{n-2} > \mu_{n-1} > 0$, then $D_\tau \subseteq D_\mu$. And in these cases, Condition (1) of Definition 3.5 holds, with (λ, μ) replaced by (ρ, τ). Assume, therefore, that $\mu_{n-2} = \mu_{n-1} > 0$, so $n-2 \in D_\tau$ and $D_\mu = D_\tau \backslash \{n-2\} \cup \{n-1\}$. Set $e = \mathrm{inc}_{\mu_{n-1}}(\mu') \geqslant 2$. For $\varpi_{n-1} : \bigoplus_{j=1}^{e} C_{\beta_{n-e}}(\mathfrak{o}) \to \bigoplus_{j=2}^{e} C_{\alpha_{n-e}}(\mathfrak{o})$ given by $(w_1, \dots, w_e) \mapsto (w_1, \dots, w_{e-1})$, the diagram commutes.

$$\begin{array}{ccc} C_\beta(\mathfrak{o}) & \xrightarrow{\vartheta^\beta_{\mu,n-1}} & \bigoplus_{j=1}^{e} C_{\beta_{n-e}}(\mathfrak{o}) \\ \downarrow \varpi & & \downarrow \varpi_{n-1} \\ C_\alpha(\mathfrak{o}) & \xrightarrow{\vartheta^\alpha_{\tau,n-2}} & \bigoplus_{j=2}^{e} C_{\alpha_{n-e}}(\mathfrak{o}) \end{array}$$

Hence, $v_{\mathfrak{p}}(\vartheta^\beta_{\mu,n-1}(w)) \geqslant \lambda_n$ implies

$$v_{\mathfrak{p}}((\varpi_{n-1}\vartheta^\beta_{\mu,n-1})(w)) = v_{\mathfrak{p}}((\vartheta^\alpha_{\tau,n-2}\varpi)(w)) \geqslant \lambda_n.$$

Since $\tau_{n-2} - \rho_{n-1} = \mu_{n-2} - \lambda_{n-1} \geqslant \mu_{n-1} - \lambda_n$, Condition (1) of Definition 3.5, again with (λ, μ) replaced by (ρ, τ), follows in this case. Because $C_{\rho,\tau} \subseteq C_{\lambda,\mu}$, Condition (2) holds as required. □

3.3. **Lattices and their intersection tableaux.** Let $\Lambda \in \mathcal{L}(V)$. Recall from Section 1.2 that $\lambda^{(i)}(\Lambda) \in \mathcal{P}_i$ is the type of $V^{(i)}/(V^{(i)} \cap \Lambda)$ for each $i \in [n]$. This yields the ***flag of partitions of intersection types*** associated with Λ

$$\lambda^\bullet(\Lambda): \quad () = \lambda^{(0)}(\Lambda) \subseteq \lambda^{(1)}(\Lambda) \subseteq \cdots \subseteq \lambda^{(n)}(\Lambda) = \lambda(\Lambda). \tag{3.7}$$

Lemma 3.8. *For all $i \in [n]$, the skew diagram $\lambda^{(i)}(\Lambda) - \lambda^{(i-1)}(\Lambda)$ is a horizontal strip.*

Proof. As $V^{(i-1)} \leqslant V^{(i)}$ with respective ranks $i-1$ and i, the quotient $\mathfrak{o}$-module $V^{(i)}/((V^{(i)} \cap \Lambda) + V^{(i-1)})$ is cyclic. Since $V^{(i)} + (V^{(i-1)} + \Lambda) = V^{(i)} + \Lambda$ and $V^{(i)} \cap (V^{(i-1)} + \Lambda) = (V^{(i)} \cap \Lambda) + V^{(i-1)}$, we have

$$V^{(i)}/((V^{(i)} \cap \Lambda) + V^{(i-1)}) \cong (V^{(i)} + \Lambda)/(V^{(i-1)} + \Lambda).$$

Since $\lambda^{(j)}(\Lambda)$ is the type of $(V^{(j)} + \Lambda)/\Lambda$ and $(V^{(i)} + \Lambda)/(V^{(i-1)} + \Lambda)$ is cyclic, the skew diagram $\lambda^{(i)}(\Lambda) - \lambda^{(i-1)}(\Lambda)$ is a horizontal strip by Lemma 2.1. □

By Lemma 3.8 the flag $\lambda^\bullet(\Lambda)$ defines a tableau $T^\bullet(\Lambda) \in \mathrm{SSYT}_n$, called the ***intersection tableau of*** Λ, as explained in Section 2.2. Given $T \in \mathrm{SSYT}_n$ we set

$$\mathcal{L}^{\mathrm{in}}_T(V) = \{\Lambda \in \mathcal{L}(V) \mid T^\bullet(\Lambda) = T\}. \tag{3.8}$$

In Theorem 4.6 we determine the cardinality $f^{\mathrm{in}}_{n,T}(\mathfrak{o}) = \#\mathcal{L}^{\mathrm{in}}_T(V)$. Thus

$$\mathrm{affS}^{\mathrm{in}}_{n,\mathfrak{o}}(\boldsymbol{Z}) = \sum_{T \in \mathrm{SSYT}_n} f^{\mathrm{in}}_{n,T}(\mathfrak{o}) \cdot \boldsymbol{Z}^{\mathrm{inc}(\lambda^\bullet(T))}. \tag{3.9}$$

Example 3.9. We consider an example in SSYT_3. Let

$$T = \begin{array}{|c|c|c|c|c|}\hline 1 & 1 & 1 & 2 & 3 \\ \hline 2 & 2 & 3 \\ \cline{1-3}\end{array} \qquad\qquad \lambda^{\bullet}(T) \;:\quad () \subseteq (3) \subseteq (4,2) \subseteq (5,3,0).$$

The lattices in $\mathcal{L}_T^{\mathrm{in}}(\mathfrak{o}^3)$ correspond to $\mathrm{GL}_3(\mathfrak{o})$-cosets of matrices of the form

$$\left\{ \begin{pmatrix} \pi^2 & a_{12} & a_{13} \\ & \pi^3 & a_{23} \\ & & \pi^3 \end{pmatrix} \in \mathrm{Mat}_3(\mathfrak{o}) \;\middle|\; \begin{array}{c} v_{\mathfrak{p}}(a_{12}) \geqslant 1,\; v_{\mathfrak{p}}(a_{23}) = 2, \\ v_{\mathfrak{p}}(a_{13}) = 0, \\ v_{\mathfrak{p}}(a_{12}a_{23} - \pi^3 a_{13}) = 3 \end{array} \right\}.$$

Thus, $f_{3,T}^{\mathrm{in}}(\mathfrak{o}) = (q-1)(q^2-q)(q^3-q^2)$, so the term in $\mathrm{affS}_{3,\mathfrak{o}}(\boldsymbol{Z})$ associated with T is

$$f_{3,T}^{\mathrm{in}}(\mathfrak{o}) \cdot \boldsymbol{Z}^{\mathrm{inc}(\lambda^{\bullet}(T))} = q^6(1-q^{-1})^3 \cdot Z_{11}^3 Z_{21}^2 Z_{22}^2 Z_{31}^2 Z_{32}^3 Z_{33}^0.$$

In Section 5 we describe a combinatorial way to obtain the factor $(1-q^{-1})^3$ in $f_{3,T}^{\mathrm{in}}(\mathfrak{o}) \cdot \boldsymbol{Z}^{\mathrm{inc}(\lambda^{\bullet}(T))}$ in terms of an invariant of a Dyck word associated with the tableau T; see Example 5.4. ♢

3.4. **Cyclic extensions of lattices by intersection data.** For a partition $\lambda \in \mathcal{P}_n$ and a lattice $\Lambda_0 \in \mathcal{L}(V^{(n-1)})$ we set

$$\mathcal{E}_{\lambda}^{\mathrm{in}}(V, \Lambda_0) = \left\{ \Lambda \in \mathcal{L}(V) \;\middle|\; \Lambda \cap V^{(n-1)} = \Lambda_0,\; \lambda(\Lambda) = \lambda \right\}.$$

The lattices in $\mathcal{E}_{\lambda}^{\mathrm{in}}(V, \Lambda_0)$ are rank-n extensions of the rank-$(n-1)$ lattice Λ_0, all of which have type λ. The formula for the cardinality of $\mathcal{E}_{\lambda}^{\mathrm{in}}(V, \Lambda_0)$ established in Theorem 3.12 plays an important role in the determination of $f_{n,T}^{\mathrm{in}}(\mathfrak{o})$, the cardinality of $\mathcal{L}_T^{\mathrm{in}}(V)$ for $T \in \mathrm{SSYT}_n$, in Theorem 4.6.

Proposition 3.10. *Set $\mu = \lambda(\Lambda_0)$. If $\mathcal{E}_{\lambda}^{\mathrm{in}}(V, \Lambda_0)$ is non-empty, then $\lambda - \mu$ is a horizontal strip.*

Proof. Suppose $\mathcal{E}_{\lambda}^{\mathrm{in}}(V, \Lambda_0) \neq \varnothing$ and $\Lambda \in \mathcal{E}_{\lambda}^{\mathrm{in}}(V, \Lambda_0)$. We have $(\Lambda + V^{(n)})/\Lambda \cong V^{(n-1)}/(\Lambda \cap V^{(n-1)}) \cong C_{\mu}(\mathfrak{o})$ and $V/\Lambda \cong C_{\lambda}(\mathfrak{o})$. Since $V/(\Lambda + V^{(n-1)})$ is cyclic, by Lemma 2.1 we conclude that $\mu \subseteq \lambda$ and $\lambda - \mu$ is a horizontal strip. □

The following proposition is the key to enumerating the lattices in $\mathcal{E}_{\lambda}^{\mathrm{in}}(V, \Lambda_0)$, and it establishes the connection between (λ, μ)-extending elements of $C_{\mu}(\mathfrak{o})$ and lattices $\Lambda \in \mathcal{L}(V)$ of type λ.

Proposition 3.11. *Let $(\lambda, \mu) \in \mathrm{HoSt}_n$ and let $\Lambda_0 \in \mathcal{L}(V^{(n-1)})$ with $\mu = \lambda(\Lambda_0)$. Let $v \in V \backslash (\mathfrak{p}V + \Lambda_0)$ and $u \in V^{(n-1)}$, and set $\Lambda = \mathfrak{o}(\pi^{|\lambda| - |\mu|} v + u) + \Lambda_0$. The following are equivalent.*

(1) $\lambda(\Lambda) = \lambda$.
(2) *The coset $u + \Lambda_0$ in $V^{(n-1)}/\Lambda_0$ is (λ, μ)-extending.*

Proof. We show that (2) implies (1). Write the nth term of $\lambda(\Lambda)$ as $\lambda_n(\Lambda)$. By Lemma 3.7, it suffices to show that $\lambda_n(\Lambda) = \lambda_n$. If $\mu_{n-1} = 0$, then $\lambda_n = 0$ since $\lambda - \mu$ is a horizontal strip. Hence, V/Λ has rank less than n. Therefore, $\lambda_n(\Lambda) = 0$ as needed, so we assume $\mu \in \mathcal{P}_{n-1} \backslash \mathcal{P}_{n-2}$.

Let $\sigma = \lambda - \mu$, and observe that $\mu_{n-1} \geqslant \lambda_n$. Note that

$$\lambda_n(\Lambda) = \min \left\{ k \in \mathbb{N}_0 \;\middle|\; \exists v_0 \in V \backslash \mathfrak{p}V,\; \pi^k v_0 \in \Lambda \right\}. \tag{3.10}$$

Let $w = u + \Lambda_0 \in V^{(n-1)}/\Lambda_0 \cong C_{\mu}(\mathfrak{o})$. Since w is (λ, μ)-extending, by Lemma 3.6 $v_{\mathfrak{p}}(w) \geqslant \lambda_n$, and therefore by (3.10),

$$\lambda_n(\Lambda) \geqslant \lambda_n.$$

Also by Lemma 3.6, we have three cases. First suppose $v_{\mathfrak{p}}(w) = \lambda_n$. Thus there exists a $u' \in V \backslash \mathfrak{p}V$ such that $\pi^{\lambda_n} u' + \Lambda_0 = w$. Set $u'' = u' + \pi^{|\sigma| - \lambda_n} v \in V \backslash \mathfrak{p}V$, so $\pi^{\lambda_n} u'' \in \Lambda$. Hence, $\lambda_n(\Lambda) \leqslant n$ in this case. Now consider the second case where

$\lambda_n = \mu_{n-1}$. This implies that $\gamma_{n-1} = \mu_{n-1} - \lambda_n = 0$. Since w is (λ,μ)-extending, $\vartheta_{\mu,n-1}(w) = 0$. Hence, there exists such a $u' \in V\backslash\mathfrak{p}V$ such that $\pi^{\mu_{n-1}}u' + \Lambda_0 = w$. As in the first case, this implies that $\lambda_n(\Lambda) \leqslant \lambda_n = \mu_{n-1}$. Now consider the final case where $\mu = (\lambda_1,\dots,\lambda_{n-1})$, so $\nu_{n-1} = |\sigma| = \lambda_n$. By Equation (3.10), $\lambda_n(\Lambda) \leqslant |\sigma| = \lambda_n$, so $\lambda_n(\Lambda) = \lambda_n$ in all three cases as required.

Now we show that (1) implies (2). Suppose w is not (λ,μ)-extending. Then there is a largest $i \in D_\mu$ such that $v_\mathfrak{p}(\vartheta_{\mu,i}(w)) < \nu_i$ or a largest $j \in I_{\lambda,\mu}$ such that $v_\mathfrak{p}(\vartheta_{\mu,j}(w)) > \nu_j$. In both cases, by applying the same inductive proof as above there is a $k \in [n]$ such that the value of $\lambda_k(\Lambda)$ is strictly smaller or larger than λ_k. (The value k depends on the value of the problematic $v_\mathfrak{p}(\vartheta_{\mu,i}(w))$ and its relation to the other ν_j.) Hence, $\lambda(\Lambda) \neq \lambda$. ☐

Theorem 3.12. *Let $(\lambda,\mu) \in \mathrm{HoSt}_n$ and let $\Lambda_0 \in \mathcal{L}(V^{(n-1)})$ of type μ. Then*

$$\#\mathcal{E}^{\mathrm{in}}_\lambda(V,\Lambda_0) = q^{\mathrm{gap}(\lambda,\mu)} \prod_{a\in J_{\lambda,\mu}} \left(1 - q^{-\mathrm{inc}_a(\mu')}\right).$$

Proof. Let $\sigma = \lambda - \mu$ be the horizontal strip. Let $v \in V\backslash(\mathfrak{p}V + V^{(n-1)})$, so $V = \mathfrak{o}v + V^{(n-1)}$. For $u \in V\backslash V^{(n-1)}$, set

$$\Lambda(u) = \mathfrak{o}u + \Lambda_0.$$

For all $u \in V$ there exist $\alpha \in \mathfrak{o}\backslash\mathfrak{p}$, $f \in \mathbb{N}_0$, and $u'' \in V^{(n-1)}$ such that

$$u + \Lambda_0 = \alpha\pi^f v + u'' + \Lambda_0. \tag{3.11}$$

By Lemma 3.8, if $\lambda(V/\Lambda(u)) = \lambda$, then $f = |\sigma|$ in (3.11). Therefore it suffices to count the number of Λ_0-cosets, $u + \Lambda_0$, in V such that $\lambda(V/\Lambda(u)) = \lambda$ as different choices of $v \in V\backslash(\mathfrak{p}V + V^{(n-1)})$ are inconsequential. Thus, by Proposition 3.11, $\#\mathcal{E}^{\mathrm{in}}_\lambda(V,\Lambda_0)$ is equal to the number of (λ,μ)-extending elements of $C_\mu(\mathfrak{o})$. For $e,m \in \mathbb{N}$ and $f \in \mathbb{N}_0$ with $f \leqslant m$, we have $\#\mathfrak{p}^f C_m(\mathfrak{o})^e = q^{e(m-f)}$. For $i \in \mathbb{N}$, set $e_i = \mathrm{inc}_{\mu_i}(\mu')$. Therefore by Proposition 3.11 and (3.1), we have

$$\begin{aligned}\#\mathcal{E}^{\mathrm{in}}_\lambda(V,\Lambda_0) &= \prod_{i\in D_\mu} q^{e_i(\mu_i-\nu_i)} \prod_{i\in I_{\lambda,\mu}} (1-q^{-e_i}) = \prod_{i\in D_\mu} q^{e_i\gamma_i} \prod_{a\in J_{\lambda,\mu}} (1-q^{-\mathrm{inc}_a(\mu')}) \\ &= q^{\mathrm{gap}(\lambda,\mu)} \prod_{a\in J_{\lambda,\mu}} (1-q^{-\mathrm{inc}_a(\mu')}).\end{aligned}$$ ☐

As consequence of Theorem 3.12 the cardinality of $\mathcal{E}^{\mathrm{in}}_\lambda(V,\Lambda_0)$ depends only on q, λ, and $\lambda(V/\Lambda_0)$. To reflect this, we define

$$\mathrm{ext}^{\mathrm{in}}_{\lambda,\mu}(\mathfrak{o}) = \#\mathcal{E}^{\mathrm{in}}_\lambda(V,\Lambda_0) \tag{3.12}$$

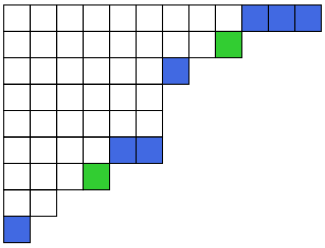
whenever $(\lambda,\mu) \in \mathrm{HoSt}_n$ and μ is the type of Λ_0.

Example 3.13. To illustrate Theorem 3.12 we compute $\mathrm{ext}^{\mathrm{in}}_{\lambda,\mu}(\mathfrak{o})$ for a pair $(\lambda,\mu) \in \mathrm{HoSt}_9$ by counting the matrices in $\mathrm{Mat}_9(\mathfrak{o})$ whose rows generate the lattices in question. Figure 3.3 displays, besides λ and μ, the horizontal strip $\sigma = \lambda - \sigma$, whose cells we color in blue, as well as the associated valuation partition ν and the gap partition γ. Cells in $C_{\lambda,\mu}$ we color in green.

We count the cosets in $\mathrm{GL}_9(\mathfrak{o})\backslash\mathrm{Mat}_9(\mathfrak{o})$ with a representative of the form

$$M = \left(\begin{array}{c|cccc} \pi^7 & \pi^4 w_2 & \pi^3 w_5 & \pi w_7 & \pi w_8 \\ \hline & \pi^9\,\mathrm{Id}_2 & & & \\ & & \pi^6\,\mathrm{Id}_3 & & \\ & & & \pi^4\,\mathrm{Id}_2 & \\ & & & & \pi^2\,\mathrm{Id}_1 \end{array}\right) \in \mathrm{Mat}_9(\mathfrak{o}),$$

where $v_\mathfrak{p}(w_2) = 0$, $v_\mathfrak{p}(w_5) \in [3]_0$, $v_\mathfrak{p}(w_7) = 0$, and $v_\mathfrak{p}(w_8) \in [1]_0$. Because we view the rows of such matrices as generating the lattices in $\mathcal{E}^{\mathrm{in}}_\lambda(\mathfrak{o}^9,\Lambda_0)$, where Λ_0 is the



$\lambda = (12, 9, 7, 6, 6, 6, 4, 2, 1) \in \mathcal{P}_9,$
$\mu = (9, 9, 6, 6, 6, 4, 4, 2) \in \mathcal{P}_8,$
$\sigma = (3, 0, 1, 0, 0, 2, 0, 0, 1) \in \mathbb{N}_0^9,$
$\nu = (4, 4, 3, 3, 3, 1, 1, 1) \in \mathcal{P}_8,$
$\gamma = (5, 5, 3, 3, 3, 3, 3, 1) \in \mathcal{P}_8.$

FIGURE 3.3. Data associated with an example $(\lambda, \nu) \in \mathrm{HoSt}_9$.

lattice generated by the lower right 8×8 matrix in M, we need only count specific cosets for the w_i, namely, $w_2 \in (\mathfrak{o}/\mathfrak{p}^5)^2$, $w_5 \in (\mathfrak{o}/\mathfrak{p}^3)^3$, $w_7 \in (\mathfrak{o}/\mathfrak{p}^3)^2$, and $w_8 \in \mathfrak{o}/\mathfrak{p}$. Their respective contributions to the number of such cosets are $q^{10}(1-q^{-2})$, q^9, $q^6(1-q^{-2})$ and q. Hence,

$$\mathrm{ext}^{\mathrm{in}}_{\lambda,\mu}(\mathfrak{o}) = q^{26}(1-q^{-2})^2.$$

Note that $26 = 5+5+3+3+3+3+3+1 = \mathrm{gap}(\lambda, \mu)$ and $J_{\lambda,\mu} = \{4, 9\}$ with

$$\mu' = (8, 8, 7, 7, 5, 5, 2, 2, 2) \in \mathcal{P}_9, \qquad \mathrm{inc}(\mu') = (0, 1, 0, \underline{2}, 0, 3, 0, 0, \underline{2}) \in \mathbb{N}_0^9. \qquad \diamondsuit$$

3.5. **Lattices and their projection tableaux.** We dualize the approach followed in Section 3.3. Recall from Section 2.3 that $V_\bullet$ is a complete isolated coflag with projections $\varpi_i : V_{(i+1)} \twoheadrightarrow V_{(i)}$. For $i \in [n]$, let $\lambda_{(i)}(\Lambda) \in \mathcal{P}_i$ be the type of $V^{(i)}/\varpi_i(\Lambda)$. This yields the ***flag of partitions of projection types*** associated with Λ, similar to (3.7):

$$\lambda_\bullet(\Lambda): \quad () = \lambda_{(0)}(\Lambda) \subseteq \lambda_{(1)}(\Lambda) \subseteq \cdots \subseteq \lambda_{(n)}(\Lambda) = \lambda(\Lambda). \tag{3.13}$$

Lemma 3.14. *For all $i \in [n]$, the skew diagram $\lambda_{(i)}(\Lambda) - \lambda_{(i-1)}(\Lambda)$ is a horizontal strip.*

Proof. For each $j \in [n-1]_0$, let $\hat{\varpi}_j = \varpi_i \varpi_{i+1} \cdots \varpi_{n-1}$, so that $\hat{\varpi}_j : V \twoheadrightarrow V_{(j)}$. Since the coimage of each ϖ_i is torsion-free, the coimage of $\hat{\varpi}_j$ is torsion-free. Let $U_{n-j} = \ker(\hat{\varpi}_j) \leqslant V$, which is therefore isolated in V, so the full pre-image of $\hat{\varpi}_j(\Lambda)$ is $\Lambda + U_{n-j}$. Hence the $U_\bullet$ form a complete isolated flag of V. For all $i \in [n]$,

$$(\Lambda + U_{n-i+1})/(\Lambda + U_{n-i}) \cong U_{n-i+1}/((U_{n-i+1} \cap \Lambda) + U_{n-i}),$$

and is therefore cyclic (see the proof of Lemma 3.8) and

$$\hat{\varpi}_i(\Lambda + U_{n-i+1})/\hat{\varpi}_i(\Lambda + U_{n-i}) \cong (\Lambda + U_{n-i+1})/(\Lambda + U_{n-i}).$$

By Lemma 2.1, the statement follows. □

By Lemma 3.14 the flag $\lambda_\bullet(\Lambda)$ determines a tableau $T_\bullet(\Lambda)$, called the ***projection tableaux of*** Λ, as explained in Section 2.2. Given $T \in \mathrm{SSYT}_n$ we set

$$\mathcal{L}^{\mathrm{pr}}_T(V) = \{\Lambda \in \mathcal{L}(V) \mid T_\bullet(\Lambda) = T\}, \tag{3.14}$$

in analogy with (3.8). In Theorem 4.6 we determine the cardinality $f^{\mathrm{pr}}_{n,T}(\mathfrak{o}) = \#\mathcal{L}^{\mathrm{pr}}_T(V)$. The following is analogous to Definition 1.5; see (3.9).

Definition 3.15. The ***affine Schubert series of projection type*** is

$$\mathrm{affS}^{\mathrm{pr}}_{n,\mathfrak{o}}(\boldsymbol{Z}) = \sum_{T \in \mathrm{SSYT}_n} f^{\mathrm{pr}}_{n,T}(\mathfrak{o}) \cdot \boldsymbol{Z}^{\mathrm{inc}(\lambda^\bullet(T))} \in \mathbb{Z}[\![\boldsymbol{Z}]\!].$$

We list these rational (!) functions for $n \leqslant 3$ in Example A.2.

Example 3.16. We revisit the tableau from Example 3.9:

$$T = \begin{array}{|c|c|c|c|c|}\hline 1&1&1&2&3\\\hline 2&2&3 \\\cline{1-3}\end{array} \qquad\qquad \lambda^\bullet(T) \;:\quad () \subseteq (3) \subseteq (4,2) \subseteq (5,3,0).$$

The lattices in $\mathcal{L}_T^{\mathrm{pr}}(\mathfrak{o}^3)$ correspond to $\mathrm{GL}_3(\mathfrak{o})$-cosets of matrices of the form

$$\left\{ \begin{pmatrix} \pi^3 & b_{12} & b_{13} \\ & \pi^3 & b_{23} \\ & & \pi^2 \end{pmatrix} \in \mathrm{Mat}_3(\mathfrak{o}) \;\middle|\; \begin{array}{c} v_{\mathfrak{p}}(b_{12}) = 2,\; v_{\mathfrak{p}}(b_{23}) \geqslant 1, \\ v_{\mathfrak{p}}(b_{13}) = 0, \\ v_{\mathfrak{p}}(b_{12}b_{23} - \pi^3 b_{13}) = 3 \end{array} \right\}.$$

Thus, $f_{3,T}^{\mathrm{pr}}(\mathfrak{o}) = (q-1)^2(q^2-q)$, so the term associated with T is

$$f_{3,T}^{\mathrm{pr}}(\mathfrak{o}) \cdot \boldsymbol{Z}^{\mathrm{inc}(\lambda^\bullet(T))} = q^4(1-q^{-1})^3 \cdot Z_{11}^3 Z_{21}^2 Z_{22}^2 Z_{31}^2 Z_{32}^3 Z_{33}^0. \qquad \diamondsuit$$

3.6. **Cyclic extensions of lattices by projection data.** We shall prove Theorem 4.6 by induction on n. For the induction step we enumerate, in Theorem 3.17, cyclic extensions of lattices. For $\lambda \in \mathcal{P}_n$ and a lattice $\Lambda_0 \in \mathcal{L}(V_{(n-1)})$, we set

$$\mathcal{E}_\lambda^{\mathrm{pr}}(V, \Lambda_0) = \{\Lambda \in \mathcal{L}(V) \mid \varpi_{n-1}(\Lambda) = \Lambda_0,\; \lambda(\Lambda) = \lambda\}.$$

In complete analogy with Proposition 3.10 one proves that the set $\mathcal{E}_\lambda^{\mathrm{pr}}(V, \Lambda_0)$ is empty unless $\lambda - \mu$ is a horizontal strip, where μ is the type of Λ_0.

Theorem 3.17. *Let $(\lambda, \mu) \in \mathrm{HoSt}_n$ and let $\Lambda_0 \in \mathcal{L}(V_{(n-1)})$ of type μ. Then*

$$\#\mathcal{E}_\lambda^{\mathrm{pr}}(V, \Lambda_0) = q^{\mathrm{gap}(\tilde{\lambda}, \tilde{\mu})} \prod_{a \in J_{\tilde{\lambda}, \tilde{\mu}}} (1 - q^{-\mathrm{inc}_a(\tilde{\mu}')}).$$

Proof. Set $\sigma = \lambda - \mu$ and $d = |\sigma|$, and let $v \in \ker(\varpi_{n-1}) \backslash \mathfrak{p}V$. By Lemma 3.14, for each lattice $\Lambda \in \mathcal{E}_\lambda^{\mathrm{pr}}(V, \Lambda_0)$, we have $\Lambda \cap \mathfrak{o}v = \mathfrak{p}^d v$. Since V is free, there exists $W \leqslant V$ such that $V = W \oplus \mathfrak{o}v$. For each $i \in [n-1]$, define $u_i + \mathfrak{p}^d v \in \mathfrak{o}v/\mathfrak{p}^d v$ such that $u_i - v_i \in W + \mathfrak{p}^d v$. Write $u = (u_1 + \mathfrak{p}^d v, \dots, u_{n-1} + \mathfrak{p}^d v) \in (\mathfrak{o}v/\mathfrak{p}^d v)^{n-1} \cong C_d(\mathfrak{o})^{n-1}$. Thus, each $\Lambda \in \mathcal{E}_\lambda^{\mathrm{pr}}(V, \Lambda_0)$ gives rise to such an element u by applying $V \mapsto V/(W + \mathfrak{p}^d v)$ to the generators of Λ. By an argument that is analogous to Proposition 3.11, we claim that these elements u both characterize such lattices and are in bijection with the (λ, μ)-extending elements of $C_d(\mathfrak{o})^{n-1}$.

It suffices to count the number of (λ, μ)-extending elements in $C_d(\mathfrak{o})^{n-1}$. Let $\nu \in \mathcal{P}_{n-1}$ be the valuation partition. Then by (3.5) and Lemma 3.3,

$$\#\mathcal{E}_\lambda^{\mathrm{pr}}(V, \Lambda_0) = \prod_{i \in D_\mu} q^{e_i(d - \nu_i)} \prod_{i \in I_{\lambda,\mu}} (1 - q^{-e_i}) = q^{\mathrm{gap}(\tilde{\lambda}, \tilde{\mu})} \prod_{a \in J_{\tilde{\lambda}, \tilde{\mu}}} (1 - q^{-\mathrm{inc}_a(\tilde{\mu}')}). \qquad \square$$

In analogy with (3.12) we define $\mathrm{ext}_{\lambda,\mu}^{\mathrm{pr}}(\mathfrak{o}) = \#\mathcal{E}_\lambda^{\mathrm{pr}}(V, \Lambda_0)$ whenever $(\lambda, \mu) \in \mathrm{HoSt}_n$ and μ is the type of Λ_0.

Example 3.18. To illustrate Theorem 3.17, we compute $\mathrm{ext}_{\lambda,\mu}^{\mathrm{pr}}(\mathfrak{o})$ for the same $(\lambda, \mu) \in \mathrm{HoSt}_9$, displayed in Figure 3.3 for which we computed $\mathrm{ext}_{\lambda,\mu}^{\mathrm{int}}(\mathfrak{o})$ in Example 3.13. Here we count the cosets in $\mathrm{GL}_9(\mathfrak{o}) \backslash \mathrm{Mat}_9(\mathfrak{o})$ with a representative of the form

$$M = \left(\begin{array}{cccc|c} \pi^9\,\mathrm{Id}_2 & & & & \pi^4 w_2 \\ & \pi^6\,\mathrm{Id}_3 & & & \pi^3 w_5 \\ & & \pi^4\,\mathrm{Id}_2 & & \pi w_7 \\ & & & \pi^2\,\mathrm{Id}_1 & \pi w_8 \\ \hline & & & & \pi^7 \end{array}\right) \in \mathrm{Mat}_9(\mathfrak{o}),$$

where $v_{\mathfrak{p}}(w_2) = 0$, $v_{\mathfrak{p}}(w_5) \in [3]_0$, $v_{\mathfrak{p}}(w_7) = 0$, and $v_{\mathfrak{p}}(w_8) \in [1]_0$. Again we need only count specific cosets for the w_i, namely, $w_2 \in (\mathfrak{o}/\mathfrak{p}^3)^2$, $w_5 \in (\mathfrak{o}/\mathfrak{p}^4)^3$, $w_7 \in (\mathfrak{o}/\mathfrak{p}^6)^2$, and $w_8 \in \mathfrak{o}/\mathfrak{p}^6$. Their respective contributions to the number of such cosets are $q^6(1-q^{-2})$, q^{12}, $q^{12}(1-q^{-2})$ and q^6. Hence,

$$\mathrm{ext}_{\lambda,\mu}^{\mathrm{pr}}(\mathfrak{o}) = q^{36}(1-q^{-2})^2.$$

Note that $\widetilde{\lambda} = (11, 10, 8, 6, 6, 6, 5, 3, 0)$ and $\widetilde{\mu} = (10, 8, 8, 6, 6, 6, 3, 3)$, so the gap partition is $(6, 6, 6, 4, 4, 4, 3, 3)$, whose entry sum is 36. $\diamondsuit$

4. Enumerating lattices by tableaux

We build off of Section 3 to enumerate lattices by their tableaux data. Here, we see the leg polynomial of Definition 1.1 come into play, and we show that it is equal to $f^{\mathrm{in}}_{n,T}(\mathfrak{o})$ and $f^{\mathrm{pr}}_{n,T}(\mathfrak{o})$, up to a monomial factor in q. At the end of this section, we prove Theorems B, C and F.

4.1. Jigsaws and complements.

We extend the jigsaw operation defined in Section 3.1 in (3.2) to tableaux. Recall from Section 2.2 that tableaux are equivalent to flags of partitions whose consecutive skew diagrams are horizontal strips.

Suppose $T \in \mathrm{SSYT}_n$ with $\lambda^{(i)} = \lambda^{(i)}(T)$ for all $i \in [n]$. Define

$$\widetilde{\lambda^{(i)}} = \left(\lambda_1^{(n)} - \lambda_i^{(i)}, \lambda_1^{(n)} - \lambda_{i-1}^{(i)}, \dots, \lambda_1^{(n)} - \lambda_1^{(i)}\right) \in \mathcal{P}_i. \tag{4.1}$$

As in Section 3.1, one argues that the skew diagram $\widetilde{\lambda^{(i+1)}} - \widetilde{\lambda^{(i)}}$ is a horizontal strip for all $i \in [n-1]$. Write $\widetilde{T}$ for the tableau obtained from the flag $\widetilde{\lambda^{\bullet}}$ defined by (4.1).

With $T = (C_1, \dots, C_\ell) \in \mathrm{SSYT}_n$, we define the ***complement tableau*** to be

$$T^{\mathrm{c}} = ([n]\backslash C_\ell,\ [n]\backslash C_{\ell-1},\ \dots,\ [n]\backslash C_1),$$

where we truncated entries equal to $\varnothing$ arising from columns $C_i = [n]$.

Proposition 4.1. *For all $T \in \mathrm{SSYT}_n$ we have $\widetilde{T} = T^{\mathrm{c}}$.*

Proof. Let $\lambda^{(k)} = \lambda^{(k)}(T) = \mathrm{sh}(T^{(k)})$. Then $\widetilde{\lambda^{(k)}}$ is obtained from the skew diagram $\rho - (\lambda_1^{(n)}, \dots, \lambda_1^{(n)}, \lambda_1^{(k)}, \dots, \lambda_k^{(k)})$ by applying a horizontal and vertical reflection, where $\rho = ((\lambda_1^{(n)})^n) \in \mathcal{P}_n$. Therefore, for $k \in [n-1]$ and all $i \in [\lambda_1^{(k)}]$,

$$(\lambda_i^{(k+1)})' + (\widetilde{\lambda^{(k+1)}})_i' = k+1, \qquad (\lambda_i^{(k)})' + (\widetilde{\lambda^{(k)}})_i' = k.$$

Hence, the conclusion holds. $\square$

Recall that $d_n(C)$ is the dimension of the Schubert variety associated with $C \subseteq [n]$; see (1.5).

Definition 4.2. For $T \in \mathrm{SSYT}_n$ and $k \in [n]$, the ***kth Schubert dimension*** and the ***kth dual Schubert dimension*** of T are

$$D_k(T) = \sum_{C \in T^{(k)}} d_k(C), \qquad D_k^{\mathrm{c}}(T) = \sum_{C \in T^{(k)}} d_k([k]\backslash C) = D_k(T^{\mathrm{c}}).$$

We now relate the gap statistic from Section 3.1 with the sum of the dimensions of the Schubert varieties associated with the columns of T.

Lemma 4.3. *For $T \in \mathrm{SSYT}_n$ and $k \in [n-1]$, we have*

$$\begin{aligned} D_{k+1}(T) &= D_k(T) + \mathrm{gap}\left(\lambda^{(k+1)}(T), \lambda^{(k)}(T)\right), \\ D_{k+1}^{\mathrm{c}}(T) &= D_k^{\mathrm{c}}(T) + \mathrm{gap}\left(\widetilde{\lambda^{(k+1)}}(T), \widetilde{\lambda^{(k)}}(T)\right). \end{aligned}$$

Proof. By Proposition 4.1, it suffices to prove the first equality. Suppose T has ℓ columns, and write $T^{(k)} = (C_1, \dots, C_\ell)$ and $T^{(k+1)} = (C_1', \dots, C_\ell')$. For each $j \in [\ell]$,

$$\begin{aligned} d_{k+1}(C_j) - d_k(C_j') = &\ \mathrm{maj}([k+1]\backslash C_j) - \mathrm{maj}([k]\backslash C_j') \\ &+ \binom{n - \#C_j'}{2} - \binom{n - \#C_j + 1}{2}. \end{aligned}$$

We have two cases to consider. First suppose $C_j = C'_j$, so $k+1$ is not contained in C_j, and $d_{k+1}(C_j) - d_k(C'_j) = \#C_j$. Now assume that $C_j = C'_j \cup \{k+1\}$. Then $d_{k+1}(C_j) - d_k(C'_j) = 0$. Therefore, $D_{k+1}(T) - D_k(T)$ is equal to the number of cells in columns without cells labeled $k+1$ whose entries are less than $k+1$. Hence,

$$D_{k+1}(T) - D_k(T) = \operatorname{gap}\left(\lambda^{(k+1)}(T), \lambda^{(k)}(T)\right). \qquad \square$$

We note that for $C \subseteq [n]$ with $k = \#C$, the sum of the Schubert dimensions $d_n(C) + d_n([n]\backslash C)$ is equal to $k(n-k)$, the dimension of the Grassmannian of k-dimensional subspaces in an n-dimensional vector space. For $\operatorname{sh}(T) = \lambda$,

$$D_n(T) + D_n^{\mathrm{c}}(T) = \sum_{i \geqslant 1} \lambda'_i(n - \lambda'_i).$$

We now express the leg polynomial of T in terms of the leg polynomial of $T^{(n-1)}$.

Lemma 4.4. *Let $T \in \mathrm{SSYT}_n$. Write $(\lambda, \mu) = (\operatorname{sh}(T), \operatorname{sh}(T^{(n-1)})) \in \mathrm{HoSt}_n$.*

$$\frac{\Phi_T(Y)}{\Phi_{T^{(n-1)}}(Y)} = \prod_{a \in J_{\lambda,\mu}} (1 - q^{-\mathrm{inc}_a(\mu')}) = \prod_{a \in J_{\tilde{\lambda},\tilde{\mu}}} (1 - q^{-\mathrm{inc}_a(\tilde{\mu}')}).$$

In particular, $\Phi_{T^{\mathrm{c}}}(Y) = \Phi_T(Y)$.

Proof. By setting $\mathscr{L}_T^{(n)} = \{(i,j) \in \mathbb{N}^2 \mid \mathrm{Leg}_T^+(i,j) \neq \varnothing,\ T_{i(j+1)} = n\}$, we have

$$\Phi_T(Y) = \Phi_{T^{(n-1)}}(Y) \prod_{(i,j) \in \mathscr{L}_T^{(n)}} \left(1 - Y^{\#\,\mathrm{Leg}_T^+(i,j)}\right).$$

Write $T = (C_1, \dots, C_\ell)$, and suppose $(i,j) \in \mathscr{L}_T^{(n)}$. Then $T_{i(j+1)} = n$ and $n \notin C_j$. Since $\lambda - \mu$ are precisely the cells of T labeled n we have $(\mu_j, j) \in C_{\lambda,\mu}$. By Lemma 3.3, the result follows. $\square$

Example 4.5. We revisit Example 3.4, where $n = 6$, $\lambda = (9,8,7,6,2,1)$, and $\mu = (9,7,7,3,2)$. Let $T \in \mathrm{SSYT}_6$ be given by the flag of partitions

$$\lambda^\bullet(T): \quad () \subseteq (4) \subseteq (6,3) \subseteq (7,6,2) \subseteq (7,7,6,2) \subseteq (9,7,7,3,2) \subseteq (9,8,7,6,2,1).$$

Figure 4.1 illustrates the conclusion of Proposition 4.1. One sees that

$$D_6(T) = 25 = 1+2+0+9+13, \qquad D_6^{\mathrm{c}}(T) = 35 = 2+5+6+12+10$$

Furthermore, $\Phi_T(Y) = (1-Y)^4(1-Y^2) = \Phi_{T^{\mathrm{c}}}(Y)$, as stated by Lemma 4.4. $\diamondsuit$

1	1	1	1	2	2	3	5	5
2	2	2	3	3	3	4	6	6
3	3	4	4	4	4	5	4	4
4	4	5	6	6	6	6	3	3
5	5	6	5	5	5	2	2	2
6	6	3	2	1	1	1	1	1

T and T^{c}

1	1	1	1	2	2	3	5	5
2	2	2	3	3	3	4	6	
3	3	4	4	4	4	5		
4	4	5	6	6	6			
5	5							
6								

T

1	1	1	1	1	2	3	6
2	2	2	5	5	5	6	
3	3	6					
4	4						
6							

T^{c}

FIGURE 4.1. An illustration of the jigsaw operation on tableaux

4.2. **Enumerating lattices by intersection and projection tableaux.** We apply the results of Section 4.1 to compute both $f^{\mathrm{in}}_{n,T}(\mathfrak{o})$ and $f^{\mathrm{pr}}_{n,T}(\mathfrak{o})$ in one motion.

Theorem 4.6. *Let $T \in \mathrm{SSYT}_n$. Then*

$$\begin{aligned} f^{\mathrm{in}}_{n,T}(\mathfrak{o}) &= q^{D_n(T)}\Phi_T(q^{-1}), \\ f^{\mathrm{pr}}_{n,T}(\mathfrak{o}) &= q^{D^{\mathrm{c}}_n(T)}\Phi_T(q^{-1}) = f^{\mathrm{in}}_{n,T^{\mathrm{c}}}(\mathfrak{o}). \end{aligned}$$

Proof. We proceed by induction on n, the case $n = 1$ being trivial since $f^{\mathrm{in}}_{1,T}(\mathfrak{o}) = f^{\mathrm{pr}}_{1,T}(\mathfrak{o}) = 1$ for $T \in \mathrm{SSYT}_1$. Assume that $n > 1$ and that the statement holds for $n-1$. Let $T \in \mathrm{SSYT}_n$, and write $(\lambda, \mu) = (\mathrm{sh}(T), \mathrm{sh}(T^{(n-1)}))$. Then we have

$$\begin{aligned} f^{\mathrm{in}}_{n,T}(\mathfrak{o}) &= f^{\mathrm{in}}_{n,T^{(n-1)}}(\mathfrak{o})\mathrm{ext}^{\mathrm{in}}_{\lambda,\mu}(\mathfrak{o}) && \text{(Equation (3.12))} \\ &= f^{\mathrm{in}}_{n,T^{(n-1)}}(\mathfrak{o})q^{\mathrm{gap}(\lambda,\mu)}\prod_{a\in J_{\lambda,\mu}}\left(1-q^{-\mathrm{inc}_a(\mu')}\right) && \text{(Theorem 3.12)} \\ &= q^{\mathrm{gap}(\lambda,\mu)+D_{n-1}(T)}\Phi_{T^{(n-1)}}(q^{-1})\prod_{a\in J_{\lambda,\mu}}\left(1-q^{-\mathrm{inc}_a(\mu')}\right) && \text{(Induction)} \\ &= q^{D_n(T)}\Phi_{T^{(n-1)}}(q^{-1})\prod_{a\in J_{\lambda,\mu}}\left(1-q^{-\mathrm{inc}_a(\mu')}\right) && \text{(Lemma 4.3)} \\ &= q^{D_n(T)}\Phi_T(q^{-1}). && \text{(Lemma 4.4)} \end{aligned}$$

Since $D^{\mathrm{c}}_k(T) = D_k(T^{\mathrm{c}})$, the statement follows by Lemma 4.4. ☐

4.3. **Proof of Theorem B and Theorem C.** We first show that, for $T \in \mathrm{SSYT}_n$,

$$\boldsymbol{Z}^{\mathrm{inc}(\lambda^{\bullet}(T))} = \prod_{C\in T}\boldsymbol{Z}_{n,C}. \tag{4.2}$$

Suppose $T = (C_1, C_2) \in \mathrm{SSYT}_n$ is a two-column tableau. Then $\lambda^{\bullet}(T) = \lambda^{\bullet}(C_1) + \lambda^{\bullet}(C_2)$, where the C_i are treated as one-column tableaux for $i \in \{1, 2\}$. Thus it suffices to show that (4.2) holds for one-column tableaux $T = (C)$ for $C \subseteq [n]$.

Let $k \in [n]$. Then $\lambda^{(k)}(T) = (1^{(r_k)}) \in \mathcal{P}_k$, where $r_k = \#(C \cap [k])$. Let $e_i \in \mathbb{N}_0^k$ be the vector with 1 in the ith entry and 0 elsewhere. Then $\mathrm{inc}(\lambda^{(k)}(T)) = e_{r_k}$ and $C(r_k) \leqslant k$. For $a = C(r_k)$ and $b = C(r_k+1)$, setting $b = n+1$ if $r_k = \#C$, we have

$$\mathrm{inc}(\lambda^{(a)}(T)) = \mathrm{inc}(\lambda^{(a+1)}(T)) = \cdots = \mathrm{inc}(\lambda^{(b-1)}(T)) = e_{r_k}.$$

Therefore, (4.2) follows from the fact that $\boldsymbol{Z}^{\mathrm{inc}(\lambda^{\bullet}(T))} = \boldsymbol{Z}_{n,C}$; see (1.6). Putting everything together, we conclude that

$$\begin{aligned} \mathrm{affS}^{\mathrm{in}}_{n,\mathfrak{o}}(\boldsymbol{Z}) &= \sum_{T\in\mathrm{SSYT}_n} f^{\mathrm{in}}_{n,T}(\mathfrak{o})\boldsymbol{Z}^{\mathrm{inc}(\lambda^{\bullet}(T))} && \text{(Equation (3.9))} \\ &= \sum_{T\in\mathrm{SSYT}_n}\Phi_T(q^{-1})q^{D_n(T)}\boldsymbol{Z}^{\mathrm{inc}(\lambda^{\bullet}(T))} && \text{(Theorem 4.6)} \\ &= \sum_{T\in\mathrm{rSSYT}_n}\Phi_T(q^{-1})\sum_{(m_C)\in\mathbb{N}^T}\prod_{C\in T}\left(q^{d_n(C)}\boldsymbol{Z}_{n,C}\right)^{m_C} && \text{(Equation (4.2))} \\ &= \sum_{T\in\mathrm{rSSYT}_n}\Phi_T(q^{-1})\prod_{C\in T}\frac{q^{d_n(C)}\boldsymbol{Z}_{n,C}}{1-q^{d_n(C)}\boldsymbol{Z}_{n,C}} \\ &= \mathsf{HLS}_n\left(q^{-1}, \left(q^{d_n(C)}\boldsymbol{Z}_{n,C}\right)_C\right) \in \mathbb{Z}[q]\,(\boldsymbol{Z}). \end{aligned}$$

This completes the proof of Theorem B. ☐

Apply Theorem 4.6 to get an analogous proof for Theorem C. ☐

4.4. **Proof of Theorem D.** We define a ring a homomorphism

$$\begin{aligned} \Upsilon_n : \mathbb{Z}[\![\boldsymbol{Z}]\!] &\to \mathbb{Z}[\![\boldsymbol{x}, \boldsymbol{y}^{\pm 1}]\!] \\ Z_{ij} &\mapsto \begin{cases} y_{n-i}^{-j} y_{n-i+1}^{j} & \text{if } i \neq n, \\ x_j y_1^j & \text{if } i = n. \end{cases} \end{aligned}$$

We first prove an intermediate equation:

$$\mathrm{HS}_{n,\mathfrak{o}}(\boldsymbol{x}, \boldsymbol{y}) = \mathrm{affS}^{\mathrm{in}}_{n,\mathfrak{o}} \left((\Upsilon_n(Z_{ij}))_{1 \leqslant j \leqslant i \leqslant n} \right). \tag{4.3}$$

Fix a lattice $\Lambda \in \mathcal{L}(\mathfrak{o}^n)$ and, for $i \in [n]$, let $\lambda^{(i)} = \lambda^{(i)}(\Lambda)$ and $\delta_i = \delta_i(\Lambda)$. By Lemma 3.8 we have, for all $i \in [n]$,

$$\delta_{n-i+1} = |\lambda^{(i)} - \lambda^{(i-1)}| = \sum_{j \geqslant 1} j \left(\mathrm{inc}_j(\lambda^{(i)}) - \mathrm{inc}_j(\lambda^{(i-1)}) \right).$$

Convening that $y_0 = 1$, this yields

$$\boldsymbol{y}^{\delta} = \prod_{i=1}^{n} y_i^{\delta_{n-i+1}} = \prod_{i=1}^{n} y_i^{\sum_{j=1}^{i} j \left(\mathrm{inc}_j(\lambda^{(i)}) - \mathrm{inc}_j(\lambda^{(i-1)}) \right)} = \prod_{1 \leqslant j \leqslant i \leqslant n} \left(\frac{y_{n-i+1}^{j}}{y_{n-i}^{j}} \right)^{\mathrm{inc}_j(\lambda^{(i)})}.$$

Thus, (4.3) holds by Equation (1.4) and since

$$\boldsymbol{x}^{\mathrm{inc}(\lambda(\Lambda))} \boldsymbol{y}^{\delta(\Lambda)} = \left(\prod_{1 \leqslant j \leqslant i \leqslant n} \left(\frac{y_{n-i+1}^{j}}{y_{n-i}^{j}} \right)^{\mathrm{inc}_j(\lambda^{(i)})} \right) \prod_{1 \leqslant j \leqslant n} x_j^{\mathrm{inc}_j(\lambda^{(n)})}.$$

The second step is to show that for all $C \subseteq [n]$, we have

$$x_{\#C} \boldsymbol{y}_{C*} = \Upsilon_n(\boldsymbol{Z}_{n,C}). \tag{4.4}$$

Set $m = \#C$. Assume that $n \in C$. In this case,

$$\begin{aligned} \Upsilon_n(\boldsymbol{Z}_{n,C}) &= x_m y_1^m \prod_{k=1}^{m-1} \prod_{\varepsilon=0}^{C(k+1)-C(k)-1} y_{n-C(k)-\varepsilon}^{-k} y_{n-C(k)-\varepsilon+1}^{k} \\ &= x_m y_1^m \prod_{k=1}^{m-1} y_{n-C(k)+1}^{k} y_{n-C(k+1)+1}^{-k} = x_m \boldsymbol{y}_{C*}. \end{aligned}$$

The case where $C \subseteq [n-1]$ is similar. Thus, (4.4) holds. Applying Theorem B and (4.4) to (4.3) completes the proof. $\square$

4.5. **Proof of Theorem F.** Write $V = V_n(\mathfrak{o}) = (V_i)_{i=1}^n$, where $V_i = \mathfrak{o}^i$. The zeta function $\zeta_V(\boldsymbol{s})$ is a sum over all finite index subrepresentations V' of V. By assumption $0 < \alpha_1(\mathfrak{o}) < \cdots < \alpha_{n-1}(\mathfrak{o}^{n-1}) < \mathfrak{o}^n$ is a complete isolated flag, so we set $V^{(i)} = \alpha_i(\mathfrak{o}^i)$.

Fix a sublattice $\Lambda \leqslant V_n$, and set $V_n' = \Lambda$. For $i \in [n-1]$, a sublattice $V_i' \leqslant \mathfrak{o}^i$ is compatible with V_n' if and only if $\alpha_i(V_i') \subseteq V^{(i)} \cap \Lambda$. Since the α_i are embeddings, Λ determines a canonical subrepresentation $V^{\Lambda} = \left(\alpha_i^{-1}(V^{(i)} \cap \Lambda)\right)_{i=1}^n$. Moreover, every family $(V_i')_{i=1}^{n-1}$ of sublattices of $V^{(i)} \cap \Lambda$ determines a subrepresentation $V' \leqslant V^{\Lambda}$ with $V_n' = \Lambda$. Since $V_i' \cong \mathfrak{o}^i$ for all $i \in [n-1]$,

$$\sum_{V_i' \leqslant V_i^{\Lambda}} \left| V_i^{\Lambda} : V_i' \right|^{-s_i} = \zeta_{\mathfrak{o}^i}(s_i).$$

Recall that $v_C = (\max(C_0 \cap [i]_0))_{i=1}^n \in \mathbb{N}^n$ for $C \subseteq [n]$. Thus, by Theorem B,

$$\sum_{\Lambda \leqslant \mathfrak{o}^n} |\mathfrak{o}^n : \Lambda|^{-s_n} \prod_{i=1}^{n-1} \left| V^{(i)} : V^{(i)} \cap \Lambda \right|^{-s_i} = \sum_{T \in \mathrm{SSYT}_n} f^{\mathrm{in}}_{n,T}(\mathfrak{o}) \prod_{i=1}^{n} q^{-\left|\lambda^{(i)}(T)\right| s_i}$$

$$= \mathrm{affS}^{\mathrm{in}}_{n,\mathfrak{o}}\left((q^{-js_i})_{1\leqslant j\leqslant i\leqslant n}\right)$$
$$= \mathsf{HLS}_n\left(q^{-1}, \left(q^{d_n(C)-v_C\cdot \boldsymbol{s}}\right)_C\right).$$

Putting everything together, we have

$$\zeta_V(\boldsymbol{s}) = \sum_{\Lambda\leqslant\mathfrak{o}^n}\prod_{i=1}^{n}\left|\mathfrak{o}^i : V_i^{\Lambda}\right|^{-s_i}\sum_{\substack{V'\leqslant V^{\Lambda}\\ V'_n=\Lambda}}\prod_{i=1}^{n-1}\left|V_i^{\Lambda}:V_i'\right|$$
$$= \mathsf{HLS}_n\left(q^{-1}, \left(q^{d_n(C)-v_C\cdot \boldsymbol{s}}\right)_C\right)\prod_{i=1}^{n-1}\zeta_{\mathfrak{o}^i}(s_i),$$

which completes the proof of Theorem F. □

5. Tableaux and Dyck word statistics

In this section we interpret the leg polynomials $\Phi_T(Y)$ from Definition 1.1 in terms of Dyck words. Write $\mathcal{D}$ for the set of finite Dyck words, viz. words in letters **0** and **1**, both with equal multiplicity, with the property that no initial segment contains more **1**s than **0**s. We define maps

$\mathsf{D} : \mathrm{SSYT}_n \to \mathcal{D}$ in Section 5.1, from tableaux to Dyck words,

$\mathsf{P} : \mathcal{D} \to \mathbb{Z}[Y]$ in Section 5.2, from Dyck words to polynomials.

Proposition 5.3 expresses the leg polynomial Φ_T in terms of $\mathsf{P}(\mathsf{D}(T))$.

5.1. **From reduced tableaux to Dyck words.** We first define D on 2-column tableaux $T = (C_1, C_2)$. Set $\overline{C_1} = C_1\backslash(C_1\cap C_2)$ and $\overline{C_2} = C_2\backslash(C_1\cap C_2)$. Clearly $a := \#\overline{C_1} \geqslant \#\overline{C_2} =: b$. We obtain a Dyck word $\mathsf{D}(T)$ as follows: form a word from the $a+b$ pairwise distinct elements in $\overline{C_1}\cup\overline{C_2}$ by writing them in natural ascending order. Now replace every element of $\overline{C_1}$ by a copy of **0** and every element of $\overline{C_2}$ by a copy of **1** and a further $a-b$ ("phantom") copies of **1**. The tableau condition ensures that $\mathsf{D}(T)$ is indeed a Dyck word of length $2a$.

To define D on a general tableaux $T = (C_1, \dots, C_\ell)$, for $\ell \in \mathbb{N}_0$, simply concatenate the Dyck words for the 2-column tableaux comprising adjacent pairs of columns of T, in natural order: $\mathsf{D}(T) = \prod_{i=1}^{\ell-1}\mathsf{D}((C_i, C_{i+1}))$.

Example 5.1. Let $C_1 = \{1,2,3,4,7,9\}$ and $C_2 = \{2,5,6,8\}$, and let $T = (C_1, C_2)$. Figure 5.1 shows, on the left, T together with the Dyck word $\mathsf{D}(T) = \mathbf{00011010\cdot 11}$ of length 10. The tableau T also yields the first two columns of the tableau T' on the right in Figure 5.1. In fact, $\mathsf{D}(T') = \mathsf{D}(T)\cdot(\mathbf{001\cdot 1})\cdot(\mathbf{0100\cdot 11})$. ♢

1	2
2	5
3	6
4	8
7	
9	

$\mathsf{D}(T) = \mathbf{0001101011}$

1	2	2	3
2	5	7	
3	6	8	
4	8		
7			
9			

$\mathsf{D}(T') = \mathbf{0001101011\cdot 0011\cdot 010011}$

Figure 5.1. Two tableaux and their Dyck words

5.2. **From Dyck words to polynomials.** Let $w \in \mathcal{D}$ be a Dyck word. There exist unique $r \in \mathbb{N}_0$ and $\ell_1, \ldots, \ell_r, m_1, \ldots, m_r \in \mathbb{N}$ such that $w = \mathbf{0}^{\ell_1}\mathbf{1}^{m_1} \ldots \mathbf{0}^{\ell_r}\mathbf{1}^{m_r}$. For $k \in [r]$, we define the kth *valley* and *peak* via

$$\mathrm{vall}_k = \sum_{i \leqslant k} (\ell_i - m_i), \qquad \mathrm{peak}_k = m_k + \mathrm{vall}_k = m_k + \sum_{i \leqslant k} (\ell_i - m_i).$$

For $m \in \mathbb{N}$, we define $[\![0]\!] = 1$, $[\![m]\!] = 1 - Y^m$, and $[\![m]\!]! = \prod_{j=1}^{m} [\![j]\!]$, and set

$$\mathsf{P} : \mathcal{D} \to \mathbb{Z}[Y], \quad w \longmapsto \prod_{k \in [r]} \frac{[\![\mathrm{peak}_k]\!]!}{[\![\mathrm{vall}_k]\!]!}. \tag{5.1}$$

One may picture the Dyck word w as a mountain range, where $\mathbf{0}$ is a line segment with positive slope and $\mathbf{1}$ is one with negative slope. Thus, w consists of r peaks at altitudes peak_k, separated by $r-1$ valleys at altitudes vall_k. Note that $\mathrm{vall}_r = 0$ by definition. The factors of the product in (5.1) correspond to the negative slopes of w, weighted by their altitudes. The mountain range in Figure 5.2, for instance, has three such segments at height 2 and one each of heights 1 and 3, whence

$$\mathsf{P}(\mathbf{0001101011}) = (1-Y)(1-Y^2)^3(1-Y^3).$$

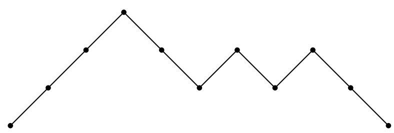


FIGURE 5.2. The Dyck word **0001101011** as a mountain range

Definition 5.2. For $T = (C_1, \ldots, C_\ell) \in \mathrm{SSYT}_n$, define the ***phantom factor***

$$\mathrm{phan}_T(Y) = \prod_{s=1}^{\ell-1} [\![(\#C_s) - (\#C_{s+1})]\!]! \in \mathbb{Z}[Y].$$

For the tableau T' on the right in Figure 5.1, for instance, we find $\mathrm{phan}_{T'}(Y) = (1-Y)^3(1-Y^2)^2$. Indeed, the three relevant column pairs yield the respective factors $[\![2]\!]!$, $[\![1]\!]!$, and $[\![2]\!]!$.

Proposition 5.3. *For all $T \in \mathrm{SSYT}_n$ we have*

$$\mathsf{P}(\mathsf{D}(T)) / \mathrm{phan}_T(Y) = \Phi_T(Y).$$

Proof. From Definition 1.1, it suffices to show that the statement holds for tableaux with at exactly two columns, so assume that $T = (C_1, C_2)$. Likewise, the leg polynomial of T is clearly oblivious of common elements of C_1 and C_2. In other words, we may also assume that $\overline{C_1} = C_1$ and $\overline{C_2} = C_2$ are disjoint sets of cardinalities $a = \#C_1 \geqslant \#C_2 = b$, say. We observe further that the leg set

$$\mathscr{L}_T = \left\{(i,1) \in \mathbb{N}^2 \mid \mathrm{Leg}_T^+(i,1) \neq \varnothing\right\} = \{k \in [b] \mid C_2(k) > C_1(k)\}$$

is in bijection with the factors defining $\mathsf{P}(\mathsf{D}(T))$ bar the final $a-b$ factors, which define $[\![\mathrm{peak}_r]\!]! = \mathrm{phan}_T(Y)$. The remaining factors correspond to the negative slopes of the mountain range associated with the Dyck word $\mathsf{D}(T) = \mathbf{0}^{\ell_1} \ldots \mathbf{1}^{m_r}$ indexed by the copies of the letter $\mathbf{1}$ outside the final factor $\mathbf{1}^{m_r}$. Each of them corresponds to a leg whose length $\#\mathrm{Leg}_T^+(i,1)$ is exactly the altitude of the corresponding negative slope. Hence $\mathsf{P}(\mathsf{D}(T)) = \Phi_T(Y)\,\mathrm{phan}_T(Y)$ as claimed. □

Example 5.4. The factor $(1-q^{-1})^3$ of $f_{3,T}^{\mathrm{in}}(\mathfrak{o})$ in Example 3.9 and of $f_{3,T}^{\mathrm{pr}}(\mathfrak{o})$ in Example 3.16 reflects the fact that $\mathsf{P}(\mathsf{D}(T)) = \mathsf{P}(\mathbf{010101}) = (1-Y)^3$. ♢

6. Reduced tableaux and Bruhat orders

In this section we portray the Hall–Littlewood–Schubert series $\mathsf{HLS}_n(Y, \boldsymbol{X})$ as a Y-analog of the fine Hilbert series of a Stanley–Reisner ring of a simplicial complex. To explain this vantage point we define, in Section 6.1, a poset structure T_n, called the *tableau order*, on the power set of $[n]$ that models adjacency of label sets of columns in tableaux and refines the set-containment relation. In Section 6.2, we show that T_n is isomorphic to a parabolic quotient of the hyperoctahedral group B_n under the Bruhat order. Its order complex $\Delta(\mathsf{T}_n)$ is isomorphic to rSSYT_n. Using this and results from Björner–Wachs [3], we prove some topological properties of $\Delta(\mathsf{T}_n)$ in Section 6.3, culminating in the proof of Theorem 1.13 in Section 6.3.1.

6.1. **Tableau order on $2^{[n]}$.** We define a partial order on $2^{[n]}$ as follows. Given non-empty $A, B \subseteq [n]$, we write $A \sqsubseteq B$ if there exists a 2-column tableau whose first column comprises the elements of A and whose second column comprises the elements of B. We write $A \sqsubseteq \varnothing$ for all $A \subseteq [n]$. We call $\sqsubseteq$ the ***tableau order*** on $2^{[n]}$, and set $\mathsf{T}_n = (2^{[n]} \backslash \{\varnothing\}, \sqsubseteq)$. We write $I \sqsubset J$ if $I \sqsubseteq J$ and $I \neq J$. We note that, for $C, D \subseteq [n]$, $C \sqsubseteq D$ if and only if $[n] \backslash D \sqsubseteq [n] \backslash C$. Figure 6.1 gives the Hasse diagrams for T_n for $n \in \{2, 3, 4\}$.

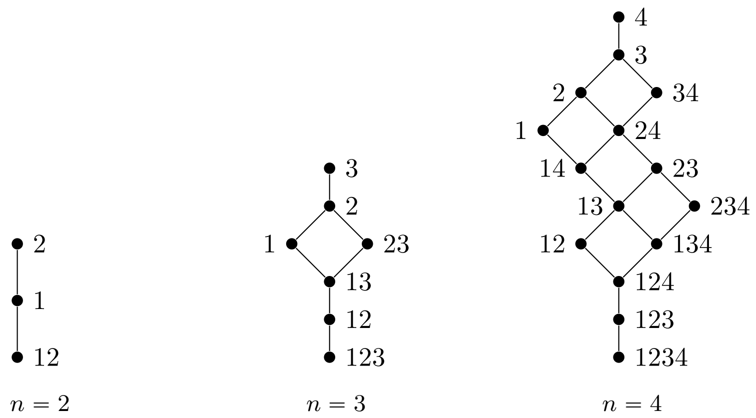


Figure 6.1. Hasse diagrams for T_n for $n \in \{2, 3, 4\}$

Let P be a poset. The ***order complex*** of P, written $\Delta(P)$, is the simplicial complex whose simplices are the flags of P.

Lemma 6.1. *The posets $\Delta(\mathsf{T}_n)$ and rSSYT_n are isomorphic.*

Proof. The columns of a tableau $T = (C_1, \dots, C_\ell) \in \mathrm{rSSYT}_n$ form, by definition of T_n, a flag $C_1 \sqsubset C_2 \sqsubset \cdots \sqsubset C_\ell$. Conversely, every such flag yields a reduced tableau $(C_1, \dots, C_\ell)$. The bijection is clearly order-preserving. □

Remark 6.2. We leave it to the reader to verify that, given $T \in \mathrm{rSSYT}$, we have $\Phi_T = 1$ if and only if the columns $(C_1, \dots, C_\ell)$ of T form a flag $C_1 \supseteq C_2 \supseteq \cdots \supseteq C_\ell$.

Let $(P, <)$ be a poset and $x, y \in P$. We say that x ***covers*** y if $x < y$ and $x \leqslant z < y$ implies $z = x$. We call such a y an ***upper cover*** for x. If $C, C' \in \mathsf{T}_n$ such that C' covers C, then we write $C \mathrel{\dot{\sqsubset}} C'$. We characterize all upper covers in T_n in Proposition 6.4.

Let $C \in \mathsf{T}_n$ and let $1 \leqslant a \leqslant b \leqslant n$ such that C contains the interval $[a, b] = \{a, a+1, \dots, b\}$. The latter is ***isolated*** in C if both $a-1$ and $b+1$ are not contained in C. For example, $C = \{1, 2, 3, 5\} \in \mathsf{T}_5$ has exactly the two isolated intervals $[1, 3]$ and $[5, 5]$. Assume that $[a, b]$ is an isolated interval in $C \in \mathsf{T}_n \backslash \{\{n\}\}$.

Note that every $C \in \mathsf{T}_n$ allows a unique decomposition

$$C = \bigsqcup_{i=1}^{k} [a_i, b_i] \tag{6.1}$$

as a disjoint sum of isolated intervals, for uniquely determined $k \in \mathbb{N}$ and $1 \leqslant a_1 \leqslant b_1 < b_1 + 1 < a_2 \leqslant b_2 < \cdots < a_k \leqslant b_k \leqslant n$. The ***elevation of*** C ***at*** $[a, b]$ is

$$\widehat{C}_{ab} = ((C \backslash \{b\}) \cup \{b+1\}) \cap [n] \in \mathsf{T}_n.$$

For $C = \{1, 2, 3, 5\} \in \mathsf{T}_5$, the two elevations are $\widehat{C}_{13} = \{1, 2, 4, 5\}$ and $\widehat{C}_{55} = \{1, 2, 3\}$.

Lemma 6.3. *Let $C = [a, b] \in \mathsf{T}_n \backslash \{\{n\}\}$. The unique upper cover for C is $\widehat{C}_{ab}$.*

Proof. Observe that $C \sqsubset \widehat{C}_{ab}$. Let $C' \subseteq [n]$ such that $C \sqsubseteq C' \sqsubset \widehat{C}_{ab}$. We have two cases depending on whether $b = n$ or not.

First we assume that $b \neq n$. We have two additional cases based on whether $\#C - \#C'$ is 0 or 1. Assume first that $\#C - \#C' = 1$. Since $C \sqsubseteq C'$ and $\min(C) = a$, it follows that $[a, \#C' + a - 1] \sqsubseteq C'$. But $\widehat{C}_{ab} \sqsubset [a, \#C' + a - 1]$, which is a contradiction. Hence, $\#C - \#C' = 0$. Let $k = \#C$, and for all $i \in [k]$,

$$C(i) \leqslant C'(i). \tag{6.2}$$

Assume via contradiction that $C \neq C'$, so there exists some $i \in [k]$ such that (6.2) is strict. It follows that $C(k) = b < b + 1 \leqslant C'(k)$. Therefore, $\widehat{C}_{ab} \sqsubseteq C'$, which is a contradiction. Thus, $C = C'$ in this case, so that $\widehat{C}_{ab}$ is the unique cover for C.

Now we assume that $b = n$. It follows that $\#C - \#C' = 1$. Since $\min(C) = a$, $[a, \#C' + a - 2] \sqsubseteq C'$, but $a = n - \#C + 1$, implying that $\widehat{C}_{ab} \sqsubseteq C'$. This is a contradiction, so $\widehat{C}_{ab}$ is the unique cover. □

Proposition 6.4. *Let $C \in \mathsf{T}_n \backslash \{\{n\}\}$ be as in (6.1). For each $i \in [k]$, we have*

$$C \mathrel{\dot{\sqsubset}} \left(\widehat{C}_{a_i, b_i} \sqcup \bigsqcup_{j \neq i} [a_j, b_j], \right)$$

and all upper covers of C in T_n are of this form.

Proof. This is a simple induction on k, the base case being Lemma 6.3. □

6.2. **Bruhat order on $B_n^{[n-1]}$.** Let (W, S) be a Coxeter system, so that W is a Coxeter group with simple reflections S. Let $\ell = \ell_W$ be the (Coxeter) length function. Let $w, w' \in W$, where w' has reduced expression $s_1 \cdots s_a$ for elements $s_i \in S$, so in particular $\ell(w') = a$. We write $w \leqslant w'$ if there exists a reduced expression $w = s_{i_1} \cdots s_{i_b}$ with $\{i_1, \dots, i_b\}_< \subseteq [a]$. The relation $\leqslant$ on (W, S) is the ***Bruhat order***; see, for instance, [3, Sec. 2.3] ("subword property").

The ***hyperoctahedral group*** B_n is a Coxeter group generated by simple reflections $S = \{s_0, \dots, s_{n-1}\}$ satisfying the relations

$$(s_0 s_1)^4 = (s_i s_{i+1})^3 = (s_j s_k)^2 = s_j^2 = 1$$

for all $i \in [n-2]$ and $j, k \in [n-1]_0$ with $|j - k| \geqslant 2$. We relations of the form $(s_j s_k)^2 = 1$ "commuting relations". The group B_n is isomorphic to the group of signed $n \times n$-permutation matrices: indeed, for $i \in [n-1]$, we may think of s_i as the matrix transposing i and $i+1$; the reflection s_0 may be represented by the diagonal matrix $\mathrm{diag}(-1, 1, 1, \dots, 1)$. In Lemma 6.6 we describe the *parabolic quotient*

$$B_n^{[n-1]} = \{w \in B_n \mid \forall i \in [n-1],\ \ell(w) < \ell(ws_i)\};$$

see [2, Lem. 2.4.3]. In [23], elements of $w \in B_n^{[n-1]}$ are called *ascending matrices* by dint of their defining property $w(1) < \cdots < w(n)$.

To this end we define elements $w_1, \dots, w_n \in B_n$ by setting $w_1 = s_0$ and $w_{k+1} = s_k w_k$ for $k \in [n-1]$. For $I = \{i_1, \dots, i_\ell\}_< \subseteq [n]$, set $w_I = w_{i_1} \cdots w_{i_\ell}$.

Lemma 6.5. *For $I \subseteq [n]$, the word w_I is a reduced expression.*

Proof. Since each simple reflection appears in the word w_i at most once, w_I is a reduced expression for all $I \subseteq [n]$ with $\#I \leqslant 1$.

Let $1 \leqslant i < j \leqslant n$. We show that we cannot apply the relation $s_k^2 = 1$, for $k \in [n]$, without increasing the length of the word $w = w_i w_j$. Since w_i is a reduced expression, by just applying commuting relations we have

$$\begin{aligned} w &= s_{i-1} \cdots s_0 s_{j-1} \cdots s_0 = s_{i-1} \cdots s_1 s_{j-1} \cdots s_2 s_0 s_1 s_0 \\ &= s_{i-1} \cdots s_2 s_{j-1} \cdots s_3 s_1 s_2 s_0 s_1 s_0 = \dots \\ &= s_{j-1} \cdots s_{i+1} (s_{i-1} s_i) \cdots (s_{k-1} s_k) \cdots (s_0 s_1) s_0. \end{aligned} \tag{6.3}$$

For all $k \in \{3, \dots, i\}$, we have reduced expressions of the form $s_{k-1} s_k s_{k-2} s_{k-1}$ and $s_0 s_1 s_0$ in (6.3). Hence, for all $I \subseteq [n]$, we cannot apply the relation $s_k^2 = 1$ without increasing the length of the expression for w_I.

Let $1 \leqslant i < j < k \leqslant n$. We show that we cannot apply $(s_r s_{r+1})^3 = 1$ without increasing the length of $w = w_i w_j w_k$. By using the commuting relations and (6.3), we have

$$\begin{aligned} w &= w_i s_{k-1} \cdots s_{j+1} (s_{j-1} s_j) \cdots (s_0 s_1) s_0 \\ &= s_{k-1} \cdots s_{j+1} (s_{j-1} s_j) \cdots (s_{i+1} s_{i+2}) w_i (s_i s_{i+1}) \cdots (s_0 s_1) s_0 \\ &= s_{k-1} \cdots s_{j+1} (s_{j-1} s_j) \cdots (s_{i+1} s_{i+2}) (s_{i-1} s_i s_{i+1}) \cdots (s_0 s_1 s_2)(s_0 s_1) s_0. \end{aligned}$$

For each $r \in \{1, \dots i-1\}$, we have expressions of the form

$$u_r := (s_r s_{r+1} s_{r+2})(s_{r-1} s_r s_{r+1})(s_{r-2} s_{r-1} s_r),$$

where $s_{-1} = 1$. We cannot apply either $(s_r s_{r+1})^3 = 1$ or $(s_{r-1} s_r)^3 = 1$ to the expression u_r without increasing its length. Hence, u_r is a reduced expression, so we cannot apply $(s_r s_{r+1})^3 = 1$ relation to w_I for all $I \subseteq [n]$ without increasing its length. The argument concerning $(s_0 s_1)^4 = 1$ is similar. □

Lemma 6.6. *We have*

$$B_n^{[n-1]} = \{w_I \mid I \subseteq [n]\}.$$

Proof. Let $k \in [n]$. Since each w_k ends with s_0, it follows that $\ell(w_k s_1) > \ell(w_k)$. For $i \in [n-1]$ and $i \geqslant k+1$, the reflection s_i commutes with all s_j for $j \in [k-1]_0$, so $\ell(w_k s_i) > \ell(w_k)$. Lastly for all $i \in [2, n-1] \cap [k]$, the reflection s_i commutes with all but at most two letters in the word $w_k = s_{k-1} \cdots s_1 s_0$, namely s_{i-1} and s_{i+1}. Thus, $\ell(w_k s_i) > \ell(w_k)$ since

$$\begin{aligned} w_k s_i = s_{k-1} \cdots s_{i+1} \underline{s_i s_{i-1} s_i} s_{i-2} \cdots s_1 s_0 &= s_{k-1} \cdots s_{i+1} \underline{s_{i-1} s_i s_{i-1}} s_{i-2} \cdots s_1 s_0 \\ &= s_{i-1} w_k. \end{aligned}$$

By Lemma 6.5, all the w_I are reduced expressions. The lemma follows. □

The restriction of the Bruhat order on B_n to $B_n^{[n-1]}$ defines a partial order. In particular, $w_1 < \cdots < w_n$. In the next proposition we relate T_n with $B_n[n-1]$ by means of the set involution $g : 2^{[n]} \to 2^{[n]}$, $I \mapsto \{n-j+1 \mid j \in [n] \backslash I\}$.

Proposition 6.7. *The map $\alpha : \mathsf{T}_n \cup \{\varnothing\} \to B_n^{[n-1]}$ given by $I \mapsto w_{g(I)}$ is an isomorphism of posets.*

Proof. Since g is a bijection and by Lemma 6.6, the map α is a bijection of sets. It remains to show that α is order-preserving. Suppose $C \sqsubseteq D$, so $[n] \backslash D \sqsubseteq [n] \backslash C$. Then there is an embedding $\iota : g(C) \hookrightarrow g(D)$ such that $\iota(x) \geqslant x$ for all $x \in g(C)$. Hence $w_{g(C)} \leqslant w_{g(D)}$, so α is an order-preserving isomorphism. □

6.3. **Combinatorial and topological properties of T_n.** Theorem 1.13 asserts that $|\Delta(\mathsf{T}_n)|$ is Cohen–Macaulay over $\mathbb{Z}$ and homeomorphic to an $\left(\binom{n+1}{2}-1\right)$-ball. We prove it in Section 6.3.1. In Section 8 we use it to describe properties of specializations of the bivariate coarsening $\mathsf{HLS}_n(Y,(X)_C)$ at special values of Y.

A finite poset P is ***graded*** if it has a unique top and bottom element and all maximal chains have the same cardinality. The ***rank*** of a finite graded poset is one less than the cardinality of a maximal chain linking bottom and top elements.

Proposition 6.8. *The poset T_n is graded of rank $\binom{n+1}{2}-1$.*

Proof. By [3, Chain Property 2.6] and Proposition 6.7, T_n is graded. Observe that

$$1 < w_1 < \cdots < w_n < w_1w_n < \cdots < w_{n-1}w_n < \cdots < w_1\cdots w_n,$$

where $1 \in B_n$ is the identity, is a maximal chain in $B_n^{[n-1]}$ with cardinality $\binom{n+1}{2}+1$. By removing the top element from $\mathsf{T}_n \cup \{\varnothing\}$, the statement follows. □

A tableau $T \in \mathrm{rSSYT}_n$ is ***maximal*** if it corresponds to a maximal flag under the isomorphism in Lemma 6.1. Recall the flag of partitions $\lambda^{\bullet}(T) = \left(\lambda^{(i)}(T)\right)_{i\in[n]}$ associated with T; see (2.2). The following result asserts that T is maximal if and only if every integer between 1 and $\binom{n+1}{2}$ is a part of some member of this flag.

Proposition 6.9. *Let $T \in \mathrm{rSSYT}_n$. Then T is maximal if and only if*

$$\left\{\lambda_j^{(i)}(T) \;\middle|\; i \in [n], j \in [i]\right\} = \left\{1,\dots,\binom{n+1}{2}\right\}. \tag{6.4}$$

Proof. Suppose (6.4) holds. Note that $\lambda_1^{(n)}(T)$ is maximal among the parts $\lambda_j^{(i)}(T)$. Since T is reduced and $\lambda_1^{(n)}(T) = \binom{n+1}{2}$, it follows that T is maximal.

Suppose T is maximal. Assume, for a contradiction, that some of the parts $\lambda_j^{(i)}$ coincide. By definition this means that there are $i, j, d \in \mathbb{N}$ such that $T_{ij} < T_{i(j+1)}$ and $T_{(i+d)j} < T_{(i+d)(j+1)}$. In other words, there exist rows i and $i+d$ whose entries in the jth column are both strictly larger than their respective neighbors on the right. Write $C = (T_{ij})_i$ for the ith column of T and $C' = (T_{i(j+1)})_i$ for its neighbor on the right. By definition we have $C \sqsubset C'$. It is easy to verify that the set $\tilde{C} = \{T_{1j},\dots,T_{ij},T_{(i+1)j}+1,T_{(i+2)j}+1,\dots\} \cap [n]$ refines this chain: $C \sqsubset \tilde{C} \sqsubset C'$. But this contradicts the maximality of T, which implies $C \overset{\bullet}{\sqsubset} C'$. So (6.4) holds. □

1	1	1	1	2	3
2	2	3			
3					

1	1	1	1	2	3
2	2	3			
3					

FIGURE 6.2. The two maximal tableaux in $\mathrm{rSSYT}_3 \cong \Delta(\mathsf{T}_3)$

To count maximal flags in $\Delta(\mathsf{T}_n)$, we consider Gelfand–Tsetlin patterns, which are known to be in bijection with tableaux. For our purposes, a ***Gelfand–Tsetlin pattern of degree*** n is a lower-triangular matrix $A = (a_{ij}) \in \mathrm{Mat}_n(\mathbb{N}_0)$, satisfying $a_{ij} \leqslant a_{(i+1)j} \leqslant a_{(i+1)(j+1)}$ for all relevant values $j \leqslant i$. We write GT_n for the set of all Gelfand–Tsetlin patterns of degree n.

The first part of the following result is well-known. It asserts that a Gelfand–Tsetlin pattern records, in its $(n-i)$th off-diagonal, the parts of the ith member the flag of partitions of a unique tableau, and all tableaux arise in this way.

Proposition 6.10. *The map*

$$\Gamma : \mathrm{SSYT}_n \longrightarrow \mathrm{GT}_n, \quad T \longmapsto \left(\lambda_{n+1-r}^{(n-r+s)}(T)\right)_{r\in[n],s\in[r]} \tag{6.5}$$

is a bijection and maps reduced maximal tableaux to Gelfand–Tsetlin patterns whose set of entries $\{a_{ij} \mid 1 \leqslant j \leqslant i \leqslant n\}$ *is* $\{1, \dots, \binom{n+1}{2}\}$. *The number of reduced maximal tableaux is*

$$\frac{\binom{n+1}{2}! \cdot \prod_{a=1}^{n-1}(a!)}{\prod_{b=1}^{n}((2b-1)!)}. \tag{6.6}$$

Proof. That Γ is a bijection follows from [22, Sec. 7.10]. The second claim follows from Proposition 6.9, and [25, Thm. 1] yields the final statement. □

The sequence defined by (6.6) is OEIS-sequence A003121 [19]. Figure 6.3 exemplifies the bijection in Proposition 6.10.

$$\begin{array}{|c|c|c|c|c|c|c|c|}\hline 1&1&1&1&2&2&3&3\\\hline 2&2&2&3\\\cline{1-4} 3&3\\\cline{1-2}\end{array} \qquad \longleftrightarrow \qquad \begin{pmatrix} 2 & & \\ 3 & 4 & \\ 5 & 7 & 9 \end{pmatrix}$$

FIGURE 6.3. A tableau in SSYT_3 and its corresponding Gelfand–Tsetlin pattern in GT_3.

For $A = (a_{ij}) \in \mathrm{GT}_n$, we define the polynomial

$$\Psi_A(Y) = \prod_{k=1}^{n}(1 - Y^k)^{d_k},$$

where d_k is defined as the number of pairs $(i, a) \in [n] \times \mathbb{N}_0$ such that a occurs k times in the $(i-1)$th off-diagonal of A and $k-1$ times in the $(i-2)$th off-diagonal of A. The polynomial $\Psi_A(Y)$ is defined in [6, (3)] and written as p_A. Observe that the set of pairs (i, a) for a fixed $k \in [n]$ are in bijection with $\{(i,j) \in \mathscr{L}_T \mid \#\,\mathrm{Leg}_T^+(i,j) = k\}$. This proves the following lemma.

Lemma 6.11. *With* $\Gamma : \mathrm{SSYT}_n \to \mathrm{GT}_n$ *as in* (6.5), *for all* $T \in \mathrm{SSYT}_n$,

$$\Phi_T(Y) = \Psi_{\Gamma(T)}(Y).$$

Feighin–Maklin establish a formula for the Hall–Littlewood polynomial $P_\lambda(\boldsymbol{x}; t)$ associated with a partition $\lambda \in \mathcal{P}_n$ in terms of the polynomials Ψ_A. We write GT_λ for the set of Gelfand–Tsetlin patterns corresponding to tableaux of shape λ. For $A \in \mathrm{GT}_\lambda$ we write $\mathrm{wt}(A) = \mathrm{wt}(\Gamma^{-1}(A)) = (\omega_1, \dots, \omega_n)$ for the weight of the corresponding tableau and set $\boldsymbol{x}^{\mathrm{wt}(A)} = x_1^{\omega_1} \cdots x^{\omega_n}$. We have ([6, Thm. 1.1])

$$P_\lambda(\boldsymbol{x}; t) = \sum_{A \in \mathrm{GT}_\lambda} \Psi_A(t)\boldsymbol{x}^{\mathrm{wt}(A)}. \tag{6.7}$$

6.3.1. *Proof of Theorem 1.13.* For $n = 1$, the statement follows since $\mathsf{T}_1 = \{\{1\}\}$, so we assume $n \geqslant 2$. By Proposition 6.7, $\mathsf{T}_n \cup \{\varnothing\} \cong B_n^{[n-1]}$, where $J = \{s_1, \dots, s_{n-1}\} \subset S$. By [3, Thm. 5.5], the Stanley–Reisner ring of $\Delta(\mathsf{T}_n)$ over $\mathbb{Z}$ is Cohen–Macaulay; hence $|\Delta(\mathsf{T}_n)|$ is Cohen–Macaulay over $\mathbb{Z}$.

For $w, w' \in B_n^{[n-1]}$, we write

$$[w, w']^J = \{z \in B_n^{[n-1]} \mid w \leqslant z \leqslant w'\}, \qquad (w, w')^J = [w, w']^J \backslash \{w, w'\}.$$

By Proposition 6.7, $\mathsf{T}_n \backslash \{[n], \{n\}\}$ is isomorphic to $(1,\ w_2 w_3 \cdots w_n)^J$. This open interval is not *full* in the sense of [3, Sec. 4] because $s_2 \in (1,\ w_2 w_3 \cdots w_n)$. Moreover

$$\ell(w_2 w_3 \cdots w_n) - \ell(1) = \binom{n+1}{2} - 1.$$

By [3, Thm. 5.4], $|\Delta(\mathsf{T}_n\backslash\{[n],\{n\}\})|$ is homeomorphic to an $\left(\binom{n+1}{2}-3\right)$-ball. Thus, $|\Delta(\mathsf{T}_n)|$ is homeomorphic to an $\left(\binom{n+1}{2}-1\right)$-ball since it is obtained from the join of $|\Delta(\mathsf{T}_n\backslash\{[n],\{n\}\})|$ by the 1-simplex.

Proposition 6.10 now completes the proof of Theorem 1.13. ∎

7. Symplectic Hecke series

In Section 7.1 we prove Theorem E, showing that the Hecke series $\mathsf{H}_{n,\mathfrak{o}}$ are instantiations of the Hall–Littlewood–Schubert series HLS_n. In Section 7.2 we apply a result of Macdonald to obtain a surprisingly simple product expression for these specific substitutions.

7.1. Proof of Theorem E.

Write $q^{-\boldsymbol{s}}$ for $(q^{-s_1},\dots,q^{-s_n})$. Then we have

$$\begin{aligned}
\mathsf{H}_{n,\mathfrak{o}}(q^{-\boldsymbol{s}},q^{N-s_0}Z) &= \hat{\tau}(s_0,s_1,\dots,s_n,Z) && \text{(Equation (1.8))}\\
&= \sum_{\lambda\in\mathcal{P}_n}\sum_{m\geqslant\lambda_1} P_\lambda\left(q^{-\boldsymbol{s}};q^{-1}\right)\left(q^{N-s_0}Z\right)^m && \text{([12, V.5. (5.2)])}\\
&= \frac{1}{1-q^{N-s_0}Z}\sum_{\lambda\in\mathcal{P}_n} P_\lambda\left(q^{-\boldsymbol{s}};q^{-1}\right)\left(q^{N-s_0}Z\right)^{\lambda_1}\\
&= \frac{1}{1-q^{N-s_0}Z}\sum_{\lambda\in\mathcal{P}_n}\sum_{A\in\mathrm{GT}_\lambda}\Psi_A(q^{-1})q^{-\operatorname{wt}(A)\boldsymbol{s}}\left(q^{N-s_0}Z\right)^{\lambda_1} && \text{(Equation (6.7))}\\
&= \frac{1}{1-q^{N-s_0}Z}\sum_{T\in\mathrm{SSYT}_n}\Phi_T(q^{-1})q^{-\operatorname{wt}(T)\boldsymbol{s}}\left(q^{N-s_0}Z\right)^{\lambda_1^{(n)}(T)} && \text{(Lemma 6.11)}\\
&= \frac{1}{1-q^{N-s_0}Z}\sum_{T\in\mathrm{rSSYT}_n}\Phi_T(q^{-1})\prod_{C\in T}\frac{q^{N-s_0-\sum_{i\in C}s_i}Z}{1-q^{N-s_0-\sum_{i\in C}s_i}Z}\\
&= \frac{1}{1-q^{N-s_0}Z}\mathsf{HLS}_n\left(q^{-1},\left(q^{N-s_0}Z\prod_{i\in C}q^{-s_i}\right)\right). && \text{(Definition 1.2)}
\end{aligned}$$

Substituting $X=q^{N-s_0}Z$ and $x_i=q^{-s_i}$ for $i\in[n]$ finishes the proof. ∎

7.2. Symplectic Hecke series at $X=1$.

The rational function $\mathsf{H}_{n,\mathfrak{o}}(\boldsymbol{x},X)$ is, for general n, very far from a product of "simple" factors; see, for instance, Example A.3 for $n=3$. The next result, essentially due to Macdonald, states that setting $X=1$ yields a neat factorization. Recall that we set $\boldsymbol{x}_C=\prod_{i\in C}x_i$ for $C\subseteq[n]$.

Proposition 7.1 (Hecke series at $X=1$)**.** *Set $x_0=1$. For all cDVR $\mathfrak{o}$ with residue field cardinality q we have*

$$\left(\mathsf{H}_{n,\mathfrak{o}}(\boldsymbol{x},X)(1-X)\right)\big|_{X=1}=\mathsf{HLS}_n\left(q^{-1},(\boldsymbol{x}_C)_C\right)=\frac{\prod_{1\leqslant i<j\leqslant n}(1-q^{-1}x_ix_j)}{\prod_{0\leqslant i<j\leqslant n}(1-x_ix_j)}.$$

Proof. For a partition λ, let $P_\lambda(\boldsymbol{x};t)$ be the Hall–Littlewood polynomial. Then

$$\begin{aligned}
\mathsf{HLS}_n(q^{-1},(\boldsymbol{x}_C)_C) &= \left(\mathsf{H}_{n,\mathfrak{o}}(\boldsymbol{x},X)(1-X)\right)\big|_{X=1} && \text{(Theorem E)}\\
&= \sum_{\lambda\in\mathcal{P}_n}P_\lambda(\boldsymbol{x};q^{-1}) && \text{([12, V.5. (5.2)])}\\
&= \frac{\prod_{1\leqslant i<j\leqslant n}(1-q^{-1}x_ix_j)}{\prod_{0\leqslant i<j\leqslant n}(1-x_ix_j)} && \text{([12, III.5. Ex. 1]).}
\end{aligned}$$

Since the equalities hold for infinitely many prime powers q, the result follows. ∎

Remark 7.2. Proposition 7.1 may be seen as a q^{-1}-analog of a classical formula known as Littlewood identity; see [12, p. 76]. We write $s_\lambda(x_1,\dots,x_n)$ for the Schur polynomial associated with the partition $\lambda \in \mathcal{P}_n$. Taking the limit $q \to \infty$ in Proposition 7.1, we obtain the following result due to Schur.

Corollary 7.3. *We have*

$$\sum_{\lambda \in \mathcal{P}_n} s_\lambda(\boldsymbol{x}) = \mathsf{HLS}_n\left(0, (\boldsymbol{x}_C)_C\right) = \sum_{T \in \mathrm{SSYT}_n} \boldsymbol{x}^{\mathrm{wt}(T)} = \prod_{0 \leqslant i < j \leqslant n} \frac{1}{1 - x_i x_j}.$$

In a similar vain, Proposition 7.1 generalizes a number of identities for generating functions enumerating tableaux T by statistics that factor over their weight $\mathrm{wt}(T)$.

8. Coarsening Hall–Littlewood–Schubert series

We write $\mathsf{HLS}_n(Y, X) = \mathsf{HLS}_n(Y, (X)_C)$ for the bivariate rational function obtained by coarsening the variables X_C to a single variable X for all non-empty $C \subseteq [n]$. In this section we focus on (fine and coarse) instances of Hall–Littlewood–Schubert series at the special values $Y = 0$, $Y = 1$, and $Y = -1$.

8.1. HLS_n at $Y = 0$.

The starting point of this section is the observation that the Hall–Littlewood–Schubert series specializes to a fine Hilbert series of a Stanley–Reisner ring. Let $\mathrm{SR}_n = \mathrm{SR}(\Delta(\mathsf{T}_n))$ be the Stanley–Reisner ring over $\mathbb{Z}$ of $\Delta(\mathsf{T}_n)$ and $\mathrm{Hilb}\left(\mathrm{SR}_n, (X_C)_C\right)$ its fine Hilbert series. A general reference to Stanley–Reisner rings and their Hilbert series is [21, Ch. II].

Lemma 8.1. *We have*

$$\mathrm{Hilb}\left(\mathrm{SR}_n, (X_C)_C\right) = \mathsf{HLS}_n\left(0, (X_C)_C\right).$$

Proof. Annihilating Y in $\mathsf{HLS}_n(Y, \boldsymbol{X})$ has the effect of effacing the leg polynomials $\Phi_T(Y)$ in the sum defining HLS_n:

$$\mathsf{HLS}_n\left(0, (X_C)_C\right) = \sum_{T \in \mathrm{rSSYT}_n} \prod_{C \in T} \frac{X_C}{1 - X_C}.$$

The claim follows from Lemma 6.1. □

We now consider $\mathsf{HLS}_n(0, X)$, the coarse Hilbert series of SR_n.

Proposition 8.2. *Write $r = \binom{n+1}{2}$. There exist $h_{n,0}, \dots, h_{n,r} \in \mathbb{N}_0$ such that*

$$\mathsf{HLS}_n(0, X) = \frac{\sum_{i=0}^{r} h_{n,i} X^i}{(1 - X)^r}, \qquad \sum_{i=0}^{r} h_{n,i} = \frac{\binom{n+1}{2}! \cdot \prod_{a=1}^{n-1}(a!)}{\prod_{b=1}^{n}((2b-1)!)}. \tag{8.1}$$

Moreover, $h_{n,0} = 1$ and $h_{n,1} = 2^n - 1 - \binom{n+1}{2}$.

Proof. By Theorem 1.13, SR_n is Cohen–Macaulay over $\mathbb{Z}$, so by [21, Cor. II.3.2] there exist $h_{n,0}, \dots, h_{n,r} \in \mathbb{N}_0$ such that

$$\mathsf{HLS}_n(0, X) = \mathrm{Hilb}\left(\mathrm{SR}_n, X\right) = \frac{\sum_{i=0}^{r} h_{n,i} X^i}{(1 - X)^r}.$$

For the second equation, note that $h_{n,0} + \dots + h_{n,r}$ is the number of maximal flags in $\Delta(\mathsf{T}_n)$; see for example [21, p. 58]. Theorem 1.13 yields their count. The last two statements follow from trivial general statements about h-vectors of simplicial complexes: T_n has cardinality $2^n - 1$ and rank $\binom{n+1}{2} - 1$; see Proposition 6.8. □

We expect that $\mathsf{HLS}_n(0, X)$ has further interesting features reflecting algebraic properties of SR_n and topological properties of $\Delta(\mathsf{T}_n)$:

Conjecture 8.3. *Use the notation of Proposition 8.2 and set $k = \binom{n-1}{2}$. Then*

(i) $h_{n,i} > 0$ for all $i \in [k]_0$ and $h_{n,k+1} = \dots = h_{n,r} = 0$;
(ii) $h_{n,i} = h_{n,k-i}$ for all $i \in [k]_0$.

Conjecture 8.3(i) suggests that the essential topological information about $\Delta(\mathsf{T}_n)$ is captured in a lower-dimensional simplicial complex than shown in Theorem 1.13. Conjecture 8.3(ii) hints at a Poincaré duality of sorts: this lower-dimensional simplicial complex is homeomorphic to a sphere and its associated Stanley–Reisner ring is Gorenstein. The following data provide evidence in favor of Conjecture 8.3. Like all other computations discussed here they were performed using `SageMath` [24].

Example 8.4. The following is the evidence we have for Conjecture 8.3:

$$\begin{aligned}
\mathsf{HLS}_1(0,X)(1-X)^1 &= 1,\\
\mathsf{HLS}_2(0,X)(1-X)^3 &= 1,\\
\mathsf{HLS}_3(0,X)(1-X)^6 &= 1+X,\\
\mathsf{HLS}_4(0,X)(1-X)^{10} &= 1+5X+5X^2+X^3,\\
\mathsf{HLS}_5(0,X)(1-X)^{15} &= 1+16X+70X^2+112X^3+70X^4+16X^5+X^6,\\
\mathsf{HLS}_6(0,X)(1-X)^{21} &= 1+42X+539X^2+2948X^3+7854X^4+10824X^5\\
&\quad +7854X^6+2948X^7+539X^8+42X^9+X^{10}.
\end{aligned}$$ ◊

8.2. HLS_n **at** $Y = 1$**.** In Section 8.1, we substituted 0 for Y, effectively effacing Φ_T for all $T \in \mathrm{SSYT}_n$. By substituting 1 for Y instead, we turn Φ_T into an indicator function, deciding whether or not the columns of T form a flag under set-containment. These flags are also known as *weak orders*. They are in bijection with the faces of the barycentric subdivision of an n-simplex.

More precisely, let Δ_n be the simplex with vertex set $[n]$ and $\mathrm{sd}(\Delta_n)$ its barycentric subdivision. The ***weak order zeta function*** (cf. [17, Def. 2.9]) is

$$\begin{aligned}
I_n^{\mathrm{wo}}((X_C)_{\varnothing\neq C\subseteq[n]}) &= \mathrm{Hilb}(\mathrm{SR}(\mathrm{sd}(\Delta_n)),(X_C)_C)\\
&= \sum_{\varnothing\neq C_1\subset\cdots\subset C_\ell\subseteq[n]}\prod_{i=1}^{\ell}\frac{X_{C_i}}{1-X_{C_i}} \in \mathbb{Q}(\boldsymbol{X}). \qquad (8.2)
\end{aligned}$$

For each $n \in \mathbb{N}$, the nth ***Eulerian polynomial*** is $\mathrm{E}_n(X) = \sum_{w\in S_n} X^{\mathrm{des}(w)}$, where $\mathrm{Des}(w) = \{i \in [n-1] \mid w(i+1) < w(i)\}$ and $\mathrm{des}(w) = \#\mathrm{Des}(w)$ and S_n is the symmetric group on $[n]$. See [15, Ch. 1].

Proposition 8.5. *We have*

$$I_n^{\mathrm{wo}}((X_C)_C) = \mathsf{HLS}_n(1,(X_C)_C).$$

In particular,

$$\frac{\mathrm{E}_n(X)}{(1-X)^n} = \mathsf{HLS}_n(1,X).$$

Proof. Let $T \in \mathrm{rSSYT}_n$. It suffices to observe that $\Phi_T(Y)$ is divisible by $1-Y$ if and only if the columns of T do not form a flag of subsets under set containment. Otherwise $\Phi_T(Y) = 1$. This follows from combining Remark 6.2 with Proposition 5.3. The second claim follows from [15, Thm. 9.1]. □

8.3. HLS_n **at** $Y = -1$**.** Perhaps the most surprising phenomena we observe occurs when we substitute Y with -1 in $\mathsf{HLS}_n(Y,X)$. Whereas with the other two substitutions to 0 and 1 we could draw upon knowledge of the simplicial complexes $\Delta(\mathsf{T}_n)$ and $\mathrm{sd}(\Delta_n)$, there does not appear to be a simplicial complex associated with the substitution to -1; see Example 8.9. Nevertheless, $\mathsf{HLS}_n(-1,X)$ seems to have a number of remarkable properties, which we record in Conjecture 8.7.

The special value $\Phi_T(-1)$ is, for any tableau T, either zero or a power of 2.

Problem 8.6. Characterize the class of tableaux T satisfying $\Phi_T(-1) = 0$.

A solution to this problem might be analogous to the partition-theoretic characterization of the vanishing of the Hall–Littlewood Q polynomials $Q_\lambda(x;-1)$ at $t=-1$ in terms of the Hall–Littlewood polynomials $P_\lambda(x;-1)$; see [12, III.8 (8.7)].

Conjecture 8.7. *Let $r=\binom{n+1}{2}$. There exist $h^-_{n,0},\dots,h^-_{n,r-1}\in\mathbb{N}_0$ such that*

$$\mathsf{HLS}_n(-1,X)=\frac{\sum_{i=0}^{r-1}h^-_{n,i}X^i}{(1-X)^r},\qquad \sum_{i=0}^{r-1}h^-_{n,i}=\frac{\binom{n}{2}!}{\prod_{i=1}^{n-1}(2i-1)^{n-i}},$$

and $h^-_{n,1}=2^n-1-\binom{n+1}{2}$. Moreover, $h^-_{n,i}=0$ if and only if $(n,i)=(2,1)$.

Remark 8.8. The sum of the $h^-_{n,i}$ in Conjecture 8.7 appears to coincide with the number of maximal chains in the poset of Dyck words of length $2(n+1)$, ordered by inclusion. Woodcock enumerated these paths in her PhD thesis and provided a bijection of these maximal paths to standard Young tableaux in staircase tableaux; see [29, Sec. 4.3 & 4.4] and OEIS-sequence A005118 [20].

Example 8.9. The following is the evidence we have for Conjecture 8.7.

$$\begin{aligned}
\mathsf{HLS}_1(-1,X)(1-X)^{1}&=1,\\
\mathsf{HLS}_2(-1,X)(1-X)^{3}&=1+X^2,\\
\mathsf{HLS}_3(-1,X)(1-X)^{6}&=1+X+6X^2+6X^3+X^4+X^5,\\
\mathsf{HLS}_4(-1,X)(1-X)^{10}&=1+5X+32X^2+120X^3+226X^4+\cdots+X^9,\\
\mathsf{HLS}_5(-1,X)(1-X)^{15}&=1+16X+179X^2+1568X^3+8545X^4\\
&\quad+30448X^5+63979X^6+83392X^7+\cdots+X^{14}.
\end{aligned}$$

By Macaulay's theorem [21, Thm. II.2.2], these polynomials are not h-polynomials of simplicial complexes, except in the case of $n=1$. ♢

9. Hall–Littlewood–Schubert series as $\mathfrak{p}$-adic integrals

In this section, we explore Hall–Littlewood–Schubert series from the perspective of $\mathfrak{p}$-adic integration. It allows us to connect these series, in Section 9.1, to integrals over $\mathfrak{p}$-adic symplectic groups and to pro-isomorphic zeta functions of nilpotent groups. We use it to simplify some of the delicate work in [1]. In Section 9.2 we use a different expression of HLS_n in terms of $\mathfrak{p}$-adic integrals associated with Igusa's local zeta function to prove Theorem A.

9.1. Symplectic integrals and zeta functions of groups.

We show that Hall–Littlewood–Schubert series specialize to $\mathfrak{p}$-adic integrals associated with symplectic groups and thus to pro-isomorphic zeta functions of nilpotent groups. We start with the former.

For a non-archimedean local field K, we denote by $\mathrm{GSp}^+_{2n}(K)$ the set of integral invertible elements in the group of general symplectic similitudes over the non-archimedean local field K. Let μ be the Haar measure on $\mathrm{GSp}^+_{2n}(K)$ such that $\mu(\mathrm{GSp}^+_{2n}(\mathfrak{o}))=1$ for the ring of integers $\mathfrak{o}$ of K, and let $|\cdot|_\mathfrak{p}$ be the $\mathfrak{p}$-adic norm. Recall $\mathrm{maj}(C)=\sum_{i\in C}i$. The next result is essentially due to Macdonald.

Theorem 9.1. *Let K be a non-archimedean local field, with ring of integers $\mathfrak{o}$ with residue field cardinality q. Then*

$$\int_{\mathrm{GSp}^+_{2n}(K)}|\det A|_\mathfrak{p}^s\,\mathrm{d}\mu=\frac{1}{1-q^{-ns}}\mathsf{HLS}_n\left(q^{-1},\left(q^{\mathrm{maj}(C)-ns}\right)_C\right).$$

Proof. The integral in the statement is equal to the zeta function $\zeta(s,\omega_{\mathrm{tr}})$ defined in [12, p. 303], where ω_{tr} is the trivial spherical function given by $(s_0,s_1,\dots,s_n)=(N,-1,-2,\dots,-n)$; see Satake's calculations [16, App. 1, 4]. Note that $\zeta(s,\omega)$, for an arbitrary spherical function ω, is defined in terms of the nth root of the determinant. By Macdonald [12, p. 304] and (1.8), we have

$$\int_{\mathrm{GSp}_{2n}^{+}(F)} |\det A|_{\mathfrak{p}}^{s}\,\mathrm{d}\mu = \hat{\tau}(N,-1,\dots,-n,q^{-ns}) = \mathsf{H}_{n,\mathfrak{o}}(q^{1},\dots,q^{n},q^{-ns}).$$

The statement follows from Theorem E. □

Now we apply Theorem 9.1 to zeta functions of groups. Write $\mathcal{H}_n$ for the n-fold centrally amalgamated product of the Heisenberg group scheme $\mathcal{H}$. Let F be a number field of degree d with ring of integers $\mathcal{O}$. For a non-zero prime ideal $\mathfrak{p} \in \mathrm{Spec}(\mathcal{O})$, we write $\mathcal{O}_{\mathfrak{p}}$ for the completion of $\mathcal{O}$ at $\mathfrak{p}$, and $F_{\mathfrak{p}}$ for the field of fractions of $\mathcal{O}_{\mathfrak{p}}$. The $\mathcal{O}$-rational points of $\mathcal{H}_n$ can be identified with the set

$$\left\{ \begin{pmatrix} 1 & u^{\mathrm{t}} & w \\ & \mathrm{Id}_n & v \\ & & 1 \end{pmatrix} \;\middle|\; u,v\in\mathcal{O}^n, w\in\mathcal{O} \right\} \subset \mathrm{GL}_{n+2}(\mathcal{O}).$$

The abstract group $\mathcal{H}_n(\mathcal{O})$ is a finitely generated nilpotent group.

Introduced in [8], the *pro-isomorphic zeta function* $\zeta_G^{\wedge}(s)$ of the finitely generated nilpotent group G enumerates subgroups $H \leqslant G$ such that H and G have the same profinite completions.

Corollary 9.2. *Let F a number field of degree d with ring of integers $\mathcal{O}$. Writing $q_{\mathfrak{p}}$ for the cardinality of the residue field of $\mathcal{O}_{\mathfrak{p}}$, we have*

$$\zeta^{\wedge}_{\mathcal{H}_n(\mathcal{O})}(s) = \prod_{(0)\neq\mathfrak{p}\in\mathrm{Spec}(\mathcal{O})} \frac{1}{1-q_{\mathfrak{p}}^{2dn-(n+1)s}} \cdot \mathsf{HLS}_n\left(q_{\mathfrak{p}}^{-1}, \left(q_{\mathfrak{p}}^{\mathrm{maj}(C)+2dn-(n+1)s}\right)_C\right).$$

Proof. Let $\mu_{\mathfrak{p}}$ be the Haar measure on $\mathrm{GSp}_{2n}^{+}(F_{\mathfrak{p}})$ such that $\mu_{\mathfrak{p}}(\mathrm{GSp}_{2n}^{+}(\mathcal{O}_{\mathfrak{p}})) = 1$ for non-zero $\mathfrak{p} \in \mathrm{Spec}(\mathcal{O})$. By [1, Prop. 5.4],

$$\zeta^{\wedge}_{\mathcal{H}_n(\mathcal{O})}(s) = \prod_{(0)\neq\mathfrak{p}\in\mathrm{Spec}(\mathcal{O})} \int_{\mathrm{GSp}_{2n}^{+}(F_{\mathfrak{p}})} |\det A|_{\mathfrak{p}}^{(1+\frac{1}{n})s-2d}\,\mathrm{d}\mu_{\mathfrak{p}}.$$

The result follows by Theorem 9.1. □

Berman, Glazer, and Schein show in [1, (42)] that the following lemma holds using essentially [1, Lem. 5.7] whose proof uses a delicate combinatorial argument. With our framework, we can simplify their argument.

Recall that S_n is the Coxeter group of type A_{n-1}, whose Coxeter length function we denote by ℓ. Recall $\mathrm{Des}(w) \subseteq [n]$ from Section 8.2 for $w \in S_n$.

Lemma 9.3. *Let K be a non-archimedean local field, with ring of integers $\mathfrak{o}$ with residue field cardinality q. For $X_i = q^{\binom{n+1}{2}-\binom{i+1}{2}-ns}$, we have*

$$\int_{\mathrm{GSp}_{2n}^{+}(K)} |\det A|_{\mathfrak{p}}^{s}\,\mathrm{d}\mu = \frac{\sum_{w\in S_n} q^{-\ell(w)} \prod_{i\in\mathrm{Des}(w)} X_i}{\prod_{i=0}^{n}(1-X_i)}.$$

Proof. It is well-known that

$$\frac{\sum_{w\in S_n} q^{-\ell(w)} \prod_{i\in\mathrm{Des}(w)} X_i}{\prod_{i=0}^{n}(1-X_i)} = \frac{1}{1-X_0}\mathsf{I}_n\left(q^{-1}, (X_i)_{i\in[n]}\right);$$

see, for instance, [18, Rem. 3.12]. Note that the Hecke series $\hat{\tau}(s_0,\dots,s_n,Z)$ is B_n-invariant (see [12, p. 302]). Therefore, by (1.8) for all $w \in S_n$ and $k \in [n]$,

$$\begin{aligned} \mathsf{H}_{n,\mathfrak{o}}(x_{w(1)},\dots,x_{w(n)},X) &= \mathsf{H}_{n,\mathfrak{o}}(\boldsymbol{x},X), \\ \mathsf{H}_{n,\mathfrak{o}}(x_1,\dots,x_{k-1},x_k^{-1},x_{k+1},\dots,x_n,x_kX) &= \mathsf{H}_{n,\mathfrak{o}}(\boldsymbol{x},X). \end{aligned} \tag{9.1}$$

Hence, we have

$$\begin{aligned}
\int_{\mathrm{GSp}_{2n}^{+}(K)} |\det A|_{\mathfrak{p}}^{s}\,\mathrm{d}\mu &= \frac{1}{1-q^{-ns}}\mathsf{HLS}_n\left(q^{-1}, \left(q^{\mathrm{maj}(C)-ns}\right)_C\right) && \text{(Theorem 9.1)}\\
&= \mathsf{H}_{n,\mathfrak{o}}(q^{1}, q^{2}, \dots, q^{n}, q^{-ns}) && \text{(Theorem E)}\\
&= \mathsf{H}_{n,\mathfrak{o}}(q^{-1}, q^{-2}, \dots, q^{-n}, q^{\binom{n+1}{2}-ns}) && \text{(Equation (9.1))}\\
&= \frac{1}{1-X_0}\mathsf{HLS}_n\left(q^{-1}, \left(q^{\mathrm{maj}([n]\setminus C)-ns}\right)_C\right) && \text{(Theorem E)}\\
&= \frac{1}{1-X_0}\mathsf{I}_n\left(q^{-1}, (X_i)_{i\in[n]}\right). && \text{(Corollary 1.9)} \quad \square
\end{aligned}$$

9.2. **Functional equations.** As before, fix $n \in \mathbb{N}$ and a cDVR $\mathfrak{o}$. Let $\boldsymbol{s} = (s_C)_{\varnothing\neq C\subseteq[n]}$ be complex variables. For $T \in \mathrm{SSYT}_n$ and $C \subseteq [n]$, we write $m_T(C) \in \mathbb{N}_0$ for the multiplicity of C as a column of T. In Lemma 9.4 we express $m_T(C)$ in terms of the parts of members of the flag of partitions $\lambda^{\bullet}(T)$ associated with T; see (2.2). These expressions, in turn, we formulate in terms of rational functions which inform the integrand of the $\mathfrak{p}$-adic integral (9.4).

Let $\boldsymbol{z} = (z_{ij})_{1\leqslant i\leqslant j\leqslant n}$ be indeterminates and let $\Xi \in \mathrm{Tr}_n(\mathbb{Z}[\boldsymbol{z}])$ be the upper-triangular $(n\times n)$-matrix with entries z_{ij} for all $1 \leqslant i \leqslant j \leqslant n$. For $k \in [n]$, write $\Xi^{(k)}$ for the lower-right $(k\times k)$-submatrix of Ξ. For each $\ell \in [n]$, let $\rho_{\ell}^{(k)} \subset \mathbb{Z}[\boldsymbol{z}]$ be the set of $\ell\times\ell$ minors of $\Xi^{(k)}$. Note that $\rho_{\ell}^{(k)}$ comprises homogeneous polynomials of degree ℓ in the variables $\{z_{ij} \mid n-k+1 \leqslant i \leqslant j \leqslant n\}$. Set $\rho_{\ell}^{(n+1)} = \rho_{\ell}^{(0)} = \{1\} = \rho_{-m}^{(k)}$ for all $\ell \in \mathbb{Z}$, $k \in \mathbb{N}$, and $m \in \mathbb{N}_0$.

Let $\mathrm{d}\mu$ be the unique normalized Haar measure on $\mathfrak{o}^{\binom{n+1}{2}} \cong \mathrm{Tr}_n(\mathfrak{o})$ such that $\mu(\mathfrak{o}^{\binom{n+1}{2}}) = 1$. For a finite set $\mathcal{X} \subset \mathfrak{o}$, let

$$v_{\mathfrak{p}}(\mathcal{X}) = \min\{v_{\mathfrak{p}}(f) : f \in \mathcal{X}\}, \qquad \|\mathcal{X}\| = \max\{|f|_{\mathfrak{p}} : f \in \mathcal{X}\} = q^{-v_{\mathfrak{p}}(\mathcal{X})},$$

where $v_{\mathfrak{p}}$ and $|\cdot|_{\mathfrak{p}}$ are the $\mathfrak{p}$-adic valuation and norm, respectively. For sets $S, S' \subseteq \mathbb{Z}[\boldsymbol{z}]$, let $S\cdot S' = \{s\cdot s' \mid s \in S, s' \in S'\}$. We use $\prod$ to denote products of several factors. Recall that, given $C \subseteq [n]$, we write $C(k)$ for the kth-smallest element of c, and that $C(\#C+1) = n+1$. Informally speaking, we extend C by the element $n+1$.

We further define four sets of polynomials in $\mathbb{Z}[\boldsymbol{z}]$:

$$R_{n,C}^{\mathrm{num}} = \bigcup_{k=1}^{\#C}\left(\rho_{C(k)-k+1}^{(C(k))} \prod_{\ell\in[\#C]\setminus\{k\}} \rho_{C(\ell)-\ell}^{(C(\ell))}\right), \qquad R_{n,C}^{\mathrm{den}} = \prod_{k=1}^{\#C} \rho_{C(k)-k}^{(C(k))},$$

$$L_{n,C}^{\mathrm{num}} = \bigcup_{k=1}^{\#C+1}\left(\rho_{C(k)-k-1}^{(C(k)-1)} \prod_{\ell\in[\#C+1]\setminus\{k\}} \rho_{C(\ell)-\ell}^{(C(\ell)-1)}\right), \quad L_{n,C}^{\mathrm{den}} = \prod_{k=1}^{\#C+1} \rho_{C(k)-k}^{(C(k)-1)}. \tag{9.2}$$

For a matrix $M \in \mathrm{Tr}_n(\mathfrak{o})$ and a set $S \subset \mathbb{Z}[\boldsymbol{z}]$, we write $S(M) \subset \mathfrak{o}$ for the evaluation of the polynomials in S evaluated at M. We fix a basis for $\mathfrak{o}^n$, so that $\Lambda(M) \leqslant \mathfrak{o}^n$ is the lattice generated by the rows of M for $M \in \mathrm{Tr}_n(\mathfrak{o})$.

Lemma 9.4. *Let $M \in \mathrm{Tr}_n(\mathfrak{o})$ be non-singular and $T = T^{\bullet}(\Lambda(M)) \in \mathrm{SSYT}_n$. For $C \in T$ we have*

$$m_T(C) = v_{\mathfrak{p}}(R_{n,C}^{\mathrm{num}}(M)) + v_{\mathfrak{p}}(L_{n,C}^{\mathrm{num}}(M)) - v_{\mathfrak{p}}(R_{n,C}^{\mathrm{den}}(M)) - v_{\mathfrak{p}}(L_{n,C}^{\mathrm{den}}(M)).$$

Proof. Let

$$r = \min\left\{\lambda_k^{(C(k))}(T) \;\middle|\; k \in [\#C]\right\},$$
$$\ell = \max\left\{\lambda_k^{(C(k)-1)}(T) \;\middle|\; k \in [\#C+1]\right\}.$$

With this terminology, the first occurrence of C as a column of T is in column $\ell+1$, the last in column r. Hence $m_T(C) = r - \ell$. For $k \in [n]$ and $i \in [n+1]$ we have

$$\lambda_i^{(k)}(T) = v_{\mathfrak{p}}(\rho_{k-i+1}^{(k)}(M)) - v_{\mathfrak{p}}(\rho_{k-i}^{(k)}(M)).$$

The lemma follows as

$$q^{-r} = \left\| \left\{ \frac{\rho_{C(k)-k+1}^{(C(k))}(M)}{\rho_{C(k)-k}^{(C(k))}(M)} \;\middle|\; k \in [\#C] \right\} \right\| = \frac{\|R_{n,C}^{\mathrm{num}}(M)\|}{\|R_{n,C}^{\mathrm{den}}(M)\|},$$

$$q^{\ell} = \left\| \left\{ \frac{\rho_{C(k)-k-1}^{(C(k)-1)}(M)}{\rho_{C(k)-k}^{(C(k)-1)}(M)} \;\middle|\; k \in [\#C+1] \right\} \right\| = \frac{\|L_{n,C}^{\mathrm{num}}(M)\|}{\|L_{n,C}^{\mathrm{den}}(M)\|}. \qquad \square$$

For $T \in \mathrm{SSYT}_n$, let

$$\mathscr{M}_{n,T}(\mathfrak{o}) = \left\{ M \in \mathrm{Tr}_n(\mathfrak{o}) \;\middle|\; \forall \varnothing \neq C \subseteq [n],\ \frac{\|L_{n,C}^{\mathrm{num}}(M)\| \, \|R_{n,C}^{\mathrm{num}}(M)\|}{\|L_{n,C}^{\mathrm{den}}(M)\| \, \|R_{n,C}^{\mathrm{den}}(M)\|} = q^{-m_T(C)} \right\}.$$

By Lemma 9.4, this is the set of matrices $M \in \mathrm{Tr}_n(\mathfrak{o})$ such that $T^{\bullet}(\Lambda(M)) = T$. We obtain a disjoint union

$$\mathrm{Tr}_n(\mathfrak{o}) = \bigsqcup_{T \in \mathrm{SSYT}_n} \mathscr{M}_{n,T}(\mathfrak{o}). \tag{9.3}$$

Recall the definition of the weight $\mathrm{wt}(T) = (\omega_1, \ldots, \omega_n)$ in Section 2.2.

Proposition 9.5. *For $T \in \mathrm{SSYT}_n$ we have*

$$\mu(\mathscr{M}_{n,T}(\mathfrak{o})) = f_{n,T}^{\mathrm{in}}(\mathfrak{o}) \cdot (1-q^{-1})^n \prod_{i=1}^{n} q^{-i\omega_{n-i+1}}.$$

Proof. We note that, for a lattice Λ with Hermite composition $(\delta_1, \ldots, \delta_n)$, we have

$$\mu\left(\{M \in \mathrm{Tr}_n(\mathfrak{o}) \mid \Lambda(M) = \Lambda\}\right) = (1-q^{-1})^n \prod_{i=1}^{n} q^{-i\delta_i}.$$

If Λ has intersection tableau $T = T^{\bullet}(\Lambda)$ of weight $(w_1, \ldots, w_n)$ and $M \in \mathrm{Tr}_n(\mathfrak{o})$ with $\Lambda(M) = \Lambda$ then $\delta_i = w_{n-i+1}$ for all $i \in [n]$. The claim follows as, by Lemma 9.4,

$$f_{n,T}^{\mathrm{in}}(\mathfrak{o}) = \frac{\mu(\mathscr{M}_{n,T}(\mathfrak{o}))}{\mu\left(\{M \in \mathrm{Tr}_n(\mathfrak{o}) \mid \Lambda(M) = \Lambda\}\right)}. \qquad \square$$

We now express $\mathsf{HLS}_n(q^{-1}, (q^{d_n(C)-s_C})_C)$ in terms of a $\mathfrak{p}$-adic integral, a key step towards our proof of Theorem A. For variables $\boldsymbol{s}' = (s_1', \ldots, s_n')$, we set

$$I_{n,\mathfrak{o}}(\boldsymbol{s}, \boldsymbol{s}') = \int_{\mathrm{Tr}_n(\mathfrak{o})} \prod_{\varnothing \neq C \subseteq [n]} \left(\frac{\|R_{n,C}^{\mathrm{num}}\| \, \|L_{n,C}^{\mathrm{num}}\|}{\|R_{n,C}^{\mathrm{den}}\| \, \|L_{n,C}^{\mathrm{den}}\|} \right)^{s_C} \prod_{i=1}^{n} |z_{ii}|^{s_i'} \, \mathrm{d}\mu. \tag{9.4}$$

Proposition 9.6. *We have*

$$\mathsf{HLS}_n(q^{-1}, (q^{d_n(C)-s_C})_C) = (1-q^{-1})^{-n} I_{n,\mathfrak{o}}(\boldsymbol{s}, -1, -2, \ldots, -n).$$

Proof. Given $\Lambda \in \mathcal{L}(\mathfrak{o}^n)$, we write $T(\Lambda) \in \mathrm{SSYT}_n$ for the tableau associated with Λ. We have

$$\begin{aligned} \mathsf{HLS}_n(q^{-1}, (q^{d_n(C)-s_C})_C) &= \sum_{\Lambda \in \mathcal{L}(\mathfrak{o}^n)} \prod_{C \in T(\Lambda)} q^{-s_C} \\ &= \sum_{T \in \mathrm{SSYT}_n} f_{n,T}^{\mathrm{in}}(\mathfrak{o}) \cdot q^{-\sum_{\varnothing \neq C \subseteq [n]} m_T(C) s_C}. \end{aligned} \tag{9.5}$$

By Lemma 3.8, the $\mathfrak{p}$-adic valuation of the diagonal elements of matrices in $\mathscr{M}_{n,T}(\mathfrak{o})$ are constant. Therefore, on each $\mathscr{M}_{n,T}(\mathfrak{o})$ the integrand of $I_{n,\mathfrak{o}}(\boldsymbol{s})$ is constant, namely

$$q^{\sum_{i=1}^n i\omega_{n-i+1} - \sum_{\varnothing\neq C\subseteq[n]} m_T(C)s_C}.$$

Using the disjoint union (9.3), Proposition 9.5, and (9.5) we obtain

$$\begin{aligned}
&(1-q^{-1})^{-n} I_{n,\mathfrak{o}}(\boldsymbol{s},-1,-2,\dots,-n)\\
&= (1-q^{-1})^{-n} \sum_{T\in\mathrm{SSYT}_n} \int_{\mathscr{M}_{n,T}(\mathfrak{o})} \prod_{\varnothing\neq C\subseteq[n]} \left(\frac{\left\|R_{n,C}^{\mathrm{num}}\right\| \left\|L_{n,C}^{\mathrm{num}}\right\|}{\left\|R_{n,C}^{\mathrm{den}}\right\| \left\|L_{n,C}^{\mathrm{den}}\right\|}\right)^{s_C} \prod_{i=1}^n |y_{ii}|^{-i}\,\mathrm{d}\mu\\
&= (1-q^{-1})^{-n} \sum_{T\in\mathrm{SSYT}_n} q^{\sum_{i=1}^n i\omega_{n-i+1} - \sum_{\varnothing\neq C\subseteq[n]} m_T(C)s_C} \mu(\mathscr{M}_{n,T}(\mathfrak{o}))\\
&= \sum_{T\in\mathrm{SSYT}_n} f_{n,T}^{\mathrm{in}}(\mathfrak{o})\cdot q^{-\sum_{\varnothing\neq C\subseteq[n]} m_T(C)s_C} = \mathsf{HLS}_n(q^{-1}, (q^{d_n(C)-s_C})_C). \qquad \square
\end{aligned}$$

Proposition 9.6 presents $\mathsf{HLS}_n(q^{-1}, (q^{d_n(C)-s_C})_C)$ in terms of the $\mathfrak{p}$-adic integral $I_{n,\mathfrak{o}}(\boldsymbol{s},\boldsymbol{s}')$, whose integrand is a product of maximal $\mathfrak{p}$-adic norms of sets of homogeneous polynomials (of the same degree for each set). For the proof of Proposition 9.8, we record the degrees of the polynomial functions involved.

Lemma 9.7. *We have*

$$\deg R_{n,C}^{\mathrm{num}} + \deg L_{n,C}^{\mathrm{num}} - \deg R_{n,C}^{\mathrm{den}} - \deg L_{n,C}^{\mathrm{den}} = \begin{cases} 1 & \text{if } C=[n],\\ 0 & \text{otherwise.}\end{cases}$$

Proof. By inspection of (9.2) we find that

$$\begin{aligned}
\deg R_{n,C}^{\mathrm{num}} &= d_n([n]\setminus C)+1, & \deg R_{n,C}^{\mathrm{den}} &= d_n([n]\setminus C),\\
\deg L_{n,C}^{\mathrm{num}} &= \max\{0, d_n([n]\setminus C)+n-\#C-1\}, & \deg L_{n,C}^{\mathrm{den}} &= d_n([n]\setminus C)+n-\#C.
\end{aligned}$$

Regarding $\deg L_{n,C}^{\mathrm{num}}$, observe that $d_n([n]\setminus C)+n-\#C-1<0$ if and only if $d_n([n]\setminus C)+n-\#C-1=-1$. The latter is equivalent to $C=[n]$. Hence,

$$\deg L_{n,[n]}^{\mathrm{num}} = \deg L_{n,[n]}^{\mathrm{den}}. \qquad \square$$

Proposition 9.8. *For all n, there exists a finite set $S=S(n)$ of primes such that*

$$I_{n,\mathfrak{o}}(\boldsymbol{s},\boldsymbol{s}')\big|_{q\to q^{-1}} = q^{-s_{[n]}-\sum_{i=1}^n s_i'} I_{n,\mathfrak{o}}(\boldsymbol{s},\boldsymbol{s}'),$$

for all cDVRs $\mathfrak{o}$ with residue characteristic not contained in S.

Proof. We use [13, Thm. 3.1] together with Lemma 9.7. The latter asserts that the degree of the rational expression associated with the variable s_C is zero unless $C=[n]$, in which case it is one. $\square$

9.3. **Proof of Theorem A.** By Proposition 9.6 it follows that, for the rings to which Proposition 9.8 applies,

$$\begin{aligned}
\mathsf{HLS}_n(q^{-1}, (q^{d_n(C)-s_C})_C)\Big|_{q\to q^{-1}} &= (-1)^n q^{-n}\cdot \left(I_{n,\mathfrak{o}}(\boldsymbol{s},-1,-2,\dots,-n)\big|_{q\to q^{-1}}\right)\\
&= (-1)^n q^{-s_{[n]}+\binom{n}{2}}\cdot \mathsf{HLS}_n(q^{-1}, (q^{d_n(C)-s_C})_C).
\end{aligned}$$

As this equation holds for infinitely many q, it follows that

$$\mathsf{HLS}_n\left(Y^{-1}, \boldsymbol{X}^{-1}\right) = (-1)^n Y^{-\binom{n}{2}} X_{[n]}\cdot \mathsf{HLS}_n\left(Y, \boldsymbol{X}\right).$$

This concludes the proof of Theorem A. $\square$

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A. Examples

We exemplify some of the paper's rational functions—all of which can be obtained as substitutions of Hall–Littlewood–Schubert series. We do this here for $n \leqslant 3$. Further data may be found at https://zenodo.org/uploads/13895162.

A.1. **Affine Schubert series.** Recall Definition 1.5 of the affine Schubert series $\mathrm{affS}^{\mathrm{in}}_{n,\mathfrak{o}}(\boldsymbol{Z})$ of intersection type.

Example A.1 (Intersection type)**.** We have

$$\mathrm{affS}^{\mathrm{in}}_{1,\mathfrak{o}}(Z_{11}) = \frac{1}{1-Z_{11}}, \quad \mathrm{affS}^{\mathrm{in}}_{2,\mathfrak{o}}(\boldsymbol{Z}) = \frac{1-Z_{11}Z_{21}}{(1-qZ_{11})(1-Z_{21})(1-Z_{11}Z_{22})}.$$

For $n=3$ we write $\mathrm{affS}^{\mathrm{in}}_{3,\mathfrak{o}}(\boldsymbol{Z}) = \mathsf{N}^{\mathrm{in}}_{3,\mathfrak{o}}(\boldsymbol{Z})/\mathsf{D}^{\mathrm{in}}_{3,\mathfrak{o}}(\boldsymbol{Z})$, where

$$\begin{aligned}
\mathsf{N}^{\mathrm{in}}_{3,\mathfrak{o}}(\boldsymbol{Z}) = 1 &- Z_{21}Z_{31}^2 - Z_{11}Z_{22}Z_{31}Z_{32} - Z_{11}Z_{21}^2Z_{32}^2 - qZ_{11}Z_{21}Z_{31}^2 \\
&+ Z_{11}Z_{21}Z_{22}Z_{31}Z_{32}^2 - qZ_{11}Z_{21}Z_{22}Z_{32}^2 + Z_{11}Z_{21}^2Z_{31}^2Z_{32} - qZ_{11}Z_{21}^2Z_{31}Z_{32} \\
&+ qZ_{11}Z_{21}Z_{22}Z_{31}^2Z_{32} + qZ_{11}Z_{21}^2Z_{31}^2Z_{32} - q^2Z_{11}Z_{21}^2Z_{31}Z_{32} + qZ_{11}Z_{21}^2Z_{31}^3 \\
&- q^2Z_{11}Z_{21}^2Z_{31}^2 + qZ_{11}^2Z_{21}Z_{22}Z_{31}Z_{32}^2 + qZ_{11}Z_{21}^3Z_{31}Z_{32}^2 - q^2Z_{11}^2Z_{21}Z_{22}Z_{32}^2 \\
&+ q^2Z_{11}Z_{21}^2Z_{31}^3 + qZ_{11}^2Z_{21}^2Z_{22}Z_{32}^3 - qZ_{11}Z_{21}^3Z_{31}^3Z_{32} + q^2Z_{11}^2Z_{21}Z_{22}Z_{31}^2Z_{32} \\
&+ q^2Z_{11}Z_{21}^3Z_{31}^2Z_{32} - qZ_{11}^2Z_{21}^2Z_{22}Z_{31}Z_{32}^3 + q^2Z_{11}^2Z_{21}^2Z_{22}Z_{32}^3 \\
&- qZ_{11}^2Z_{21}^2Z_{22}Z_{31}^2Z_{32}^2 + q^2Z_{11}^2Z_{21}^2Z_{22}Z_{31}Z_{32}^2 + q^2Z_{11}^2Z_{21}^3Z_{31}Z_{32}^2 \\
&- q^2Z_{11}^2Z_{21}^2Z_{22}Z_{31}^2Z_{32}^2 + q^3Z_{11}^2Z_{21}^2Z_{22}Z_{31}Z_{32}^2 - q^2Z_{11}^2Z_{21}^3Z_{31}^3Z_{32} \\
&+ q^3Z_{11}^2Z_{21}^3Z_{31}^2Z_{32} - q^2Z_{11}^2Z_{21}^3Z_{22}Z_{31}Z_{32}^3 - q^3Z_{11}^2Z_{21}^2Z_{22}Z_{31}^3Z_{32} \\
&- q^3Z_{11}^2Z_{21}^4Z_{31}^2Z_{32}^2 - q^3Z_{11}^3Z_{21}^3Z_{22}Z_{31}Z_{32}^3 + q^3Z_{11}^3Z_{21}^4Z_{22}Z_{31}^3Z_{32}^3, \\
\mathsf{D}^{\mathrm{in}}_{3,\mathfrak{o}}(\boldsymbol{Z}) = {} &(1-Z_{31})(1-Z_{21}Z_{32})(1-Z_{11}Z_{22}Z_{33})(1-qZ_{21}Z_{31})(1-qZ_{11}Z_{21}Z_{32}) \\
&\times (1-q^2Z_{11}Z_{22}Z_{32})(1-q^2Z_{11}Z_{21}Z_{31}).
\end{aligned}$$

Recall Definition 3.15 of the affine Schubert series $\mathrm{affS}^{\mathrm{pr}}_{n,\mathfrak{o}}(\boldsymbol{Z})$ of projection type.

Example A.2 (Projection type)**.** We have

$$\mathrm{affS}^{\mathrm{pr}}_{1,\mathfrak{o}}(Z_{11}) = \frac{1}{1-Z_{11}}, \quad \mathrm{affS}^{\mathrm{pr}}_{2,\mathfrak{o}}(\boldsymbol{Z}) = \frac{1-Z_{11}Z_{21}}{(1-Z_{11})(1-qZ_{21})(1-Z_{11}Z_{22})}.$$

For $n=3$ we write $\mathrm{affS}^{\mathrm{pr}}_{n,\mathfrak{o}}(\boldsymbol{Z}) = \mathsf{N}^{\mathrm{pr}}_{3,\mathfrak{o}}(\boldsymbol{Z})/\mathsf{D}^{\mathrm{pr}}_{3,\mathfrak{o}}(\boldsymbol{Z})$, where

$$\begin{aligned}
\mathsf{N}^{\mathrm{pr}}_{3,\mathfrak{o}}(\boldsymbol{Z}) = 1 &- Z_{11}Z_{22}Z_{31}Z_{32} - Z_{11}Z_{21}^2Z_{31}^2 - qZ_{11}Z_{21}Z_{31}^2 - q^2Z_{21}Z_{31}^2 \\
&- Z_{11}^2Z_{21}Z_{22}Z_{32}^2 - qZ_{11}Z_{21}Z_{22}Z_{32}^2 - qZ_{11}Z_{21}^2Z_{31}Z_{32} - q^2Z_{11}Z_{21}^2Z_{32}^2 \\
&+ Z_{11}^2Z_{21}Z_{22}Z_{31}^2Z_{32} + qZ_{11}Z_{21}Z_{22}Z_{31}^2Z_{32} - q^2Z_{11}Z_{21}^2Z_{31}Z_{32} + qZ_{11}Z_{21}^2Z_{31}^3 \\
&+ Z_{11}^2Z_{21}^2Z_{22}Z_{31}Z_{32}^2 + qZ_{11}^2Z_{21}Z_{22}Z_{31}Z_{32}^2 + q^2Z_{11}Z_{21}Z_{22}Z_{31}Z_{32}^2 \\
&+ q^2Z_{11}Z_{21}^2Z_{31}^2Z_{32} + q^2Z_{11}Z_{21}^2Z_{31}^3 + qZ_{11}^2Z_{21}^2Z_{22}Z_{32}^3 + qZ_{11}^2Z_{21}^2Z_{22}Z_{31}Z_{32}^2 \\
&+ qZ_{11}^2Z_{21}^3Z_{31}^2Z_{32} + q^2Z_{11}Z_{21}^3Z_{31}^2Z_{32} + q^3Z_{11}Z_{21}^2Z_{31}^2Z_{32} + q^2Z_{11}^2Z_{21}^2Z_{22}Z_{32}^3 \\
&- qZ_{11}^2Z_{21}^2Z_{22}Z_{31}^2Z_{32}^2 + q^2Z_{11}^2Z_{21}^3Z_{31}Z_{32}^2 + q^3Z_{11}Z_{21}^3Z_{31}Z_{32}^2 \\
&- qZ_{11}^2Z_{21}^2Z_{22}Z_{31}^3Z_{32} - q^2Z_{11}^2Z_{21}^2Z_{22}Z_{31}^2Z_{32}^2 - q^2Z_{11}^2Z_{21}^3Z_{31}^3Z_{32} \\
&- q^3Z_{11}Z_{21}^3Z_{31}^3Z_{32} - qZ_{11}^3Z_{21}^3Z_{22}Z_{31}Z_{32}^3 - q^2Z_{11}^2Z_{21}^3Z_{22}Z_{31}Z_{32}^3
\end{aligned}$$

$$
\begin{aligned}
&- q^3 Z_{11}^2 Z_{21}^2 Z_{22} Z_{31} Z_{32}^3 - q^3 Z_{11}^2 Z_{21}^4 Z_{31}^2 Z_{32}^2 + q^3 Z_{11}^3 Z_{21}^4 Z_{22} Z_{31}^3 Z_{32}^3,\\
\mathsf{D}^{\mathrm{pr}}_{3,\mathfrak{o}}(\boldsymbol{Z}) &= (1 - Z_{11}Z_{22}Z_{33})(1 - Z_{11}Z_{22}Z_{32})(1 - Z_{11}Z_{21}Z_{31})(1 - qZ_{21}Z_{31})(1 - q^2 Z_{31})\\
&\times (1 - qZ_{11}Z_{21}Z_{32})(1 - q^2 Z_{21}Z_{32}).
\end{aligned}
$$

A.2. **Symplectic Hecke series.** Recall the polynomials $\mathsf{H}^{\mathrm{num}}_n(Y, \boldsymbol{x}, X)$ yielding the numerators of the Hecke series $\mathsf{H}_{n,\mathfrak{o}}(\boldsymbol{x}, X)$; see (1.7).

Example A.3.

$$
\begin{aligned}
\mathsf{H}^{\mathrm{num}}_1(Y, \boldsymbol{x}, X) &= 1,\\
\mathsf{H}^{\mathrm{num}}_2(Y, \boldsymbol{x}, X) &= 1 - Yx_1x_2X^2,\\
\mathsf{H}^{\mathrm{num}}_3(Y, \boldsymbol{x}, X) &= 1 - x_1x_2x_3X^2 - Yx_2x_3X^2 - Yx_1x_3X^2 - Yx_1x_2X^2 - Yx_1x_2x_3X^2\\
&+ Yx_1x_2x_3X^3 - Yx_1x_2x_3^2X^2 - Yx_1x_2^2x_3X^2 - Yx_1^2x_2x_3X^2\\
&- Y^2x_1x_2x_3X^2 + Yx_1x_2x_3^2X^3 + Yx_1x_2^2x_3X^3 + Yx_1^2x_2x_3X^3\\
&+ Y^2x_1x_2x_3X^3 + Yx_1x_2^2x_3^2X^3 + Yx_1^2x_2x_3^2X^3 + Y^2x_1x_2x_3^2X^3\\
&+ Yx_1^2x_2^2x_3X^3 + Y^2x_1x_2^2x_3X^3 + Y^2x_1^2x_2x_3X^3 + Yx_1^2x_2^2x_3^2X^3\\
&+ Y^2x_1x_2^2x_3^2X^3 + Y^2x_1^2x_2x_3^2X^3 + Y^2x_1^2x_2^2x_3X^3 - Yx_1^2x_2^2x_3^2X^4\\
&- Y^2x_1x_2^2x_3^2X^4 - Y^2x_1^2x_2x_3^2X^4 - Y^2x_1^2x_2^2x_3X^4 + Y^2x_1^2x_2^2x_3^2X^3\\
&- Y^2x_1^2x_2^2x_3^2X^4 - Y^2x_1^2x_2^2x_3^3X^4 - Y^2x_1^2x_2^3x_3^2X^4 - Y^2x_1^3x_2^2x_3^2X^4\\
&- Y^3x_1^2x_2^2x_3^2X^4 + Y^3x_1^3x_2^3x_3^3X^6.
\end{aligned}
$$

A.3. **Hermite–Smith series.** Recall Definition 1.8 of the the Hermite–Smith series $\mathrm{HS}_{n,\mathfrak{o}}(\boldsymbol{x}, \boldsymbol{y})$.

Example A.4. We have

$$
\mathrm{HS}_{1,\mathfrak{o}}(x_1, y_1) = \frac{1}{1 - x_1y_1}, \quad \mathrm{HS}_{2,\mathfrak{o}}(\boldsymbol{x}, \boldsymbol{y}) = \frac{1 - x_1^2y_1y_2}{(1 - x_1y_1)(1 - x_2y_1y_2)(1 - qx_1y_2)}.
$$

For $n = 3$ we write $\mathrm{HS}_{3,\mathfrak{o}}(\boldsymbol{x}, \boldsymbol{y}) = \mathrm{HS}^{\mathrm{num}}_{3,\mathfrak{o}}(\boldsymbol{x}, \boldsymbol{y}) / \mathrm{HS}^{\mathrm{den}}_{3,\mathfrak{o}}(\boldsymbol{x}, \boldsymbol{y})$, where

$$
\begin{aligned}
\mathrm{HS}^{\mathrm{num}}_{3,\mathfrak{o}}(\boldsymbol{x}, \boldsymbol{y}) &= 1 - x_1^2y_1y_2 - x_1x_2y_1y_2y_3 - qx_1^2y_1y_3 - x_2^2y_1^2y_2y_3 - qx_1x_2y_1y_2y_3\\
&- q^2x_1^2y_2y_3 - qx_2^2y_1y_2^2y_3 + x_1^2x_2y_1^2y_2y_3 - q^2x_1x_2y_1y_2y_3 + qx_1^3y_1y_2y_3\\
&- q^2x_2^2y_1y_2y_3^2 + x_1x_2^2y_1^2y_2^2y_3 + qx_1^2x_2y_1y_2^2y_3 + qx_1^2x_2y_1^2y_2y_3\\
&+ q^2x_1^3y_1y_2y_3 + qx_1x_2^2y_1^2y_2y_3^2 + q^2x_1^2x_2y_1y_2y_3^2 + qx_1x_2^2y_1^2y_2^2y_3\\
&+ q^2x_1^2x_2y_1y_2^2y_3 + qx_2^3y_1^2y_2^2y_3^2 + q^2x_1x_2^2y_1y_2^2y_3^2 + q^2x_1x_2^2y_1^2y_2y_3^2\\
&+ q^3x_1^2x_2y_1y_2y_3^2 - qx_1^3x_2y_1^2y_2^2y_3 + q^2x_2^3y_1^2y_2^2y_3^2 - qx_1^2x_2^2y_1^2y_2^2y_3^2\\
&+ q^3x_1x_2^2y_1y_2^2y_3^2 - q^2x_1^3x_2y_1^2y_2y_3^2 - qx_1x_2^3y_1^3y_2^2y_3^2 - q^2x_1^2x_2^2y_1^2y_2^2y_3^2\\
&- q^3x_1^3x_2y_1y_2^2y_3^2 - q^2x_1x_2^3y_1^2y_2^3y_3^2 - q^3x_1^2x_2^2y_1^2y_2^2y_3^2 - q^3x_1x_2^3y_1^2y_2^2y_3^3\\
&+ q^3x_1^3x_2^3y_1^3y_2^3y_3^3,\\
\mathrm{HS}^{\mathrm{den}}_{3,\mathfrak{o}}(\boldsymbol{x}, \boldsymbol{y}) &= (1 - x_1y_1)(1 - x_2y_1y_2)(1 - qx_1y_2)(1 - x_3y_1y_2y_3)(1 - qx_2y_1y_3)\\
&\times (1 - q^2x_1y_3)(1 - q^2x_2y_2y_3).
\end{aligned}
$$

A.4. **Quiver representation zeta functions.** Recall the definition (1.9) of the zeta function $\zeta_{V_n(\mathfrak{o})}(\boldsymbol{s})$ of the $\mathfrak{o}$-representation $V_n(\mathfrak{o})$ of the dual star quiver S^*_n.

Example A.5. Set $t_i = q^{-s_i}$ for $i \in [n]$. We have

$$
\zeta_{V_1(\mathfrak{o})}(s_1) = \frac{1}{1 - t_1}, \qquad \zeta_{V_2(\mathfrak{o})}(\boldsymbol{s}) = \frac{1 - t_1t_2^3}{(1 - t_2)(1 - t_2^2)(1 - t_1t_2^2)(1 - qt_1t_2)}.
$$

For $n = 3$ we write $\zeta_{V_3(\mathfrak{o})}(\boldsymbol{s}) = Z^{\mathrm{num}}_{V_3(\mathfrak{o})}(\boldsymbol{s})/Z^{\mathrm{den}}_{V_3(\mathfrak{o})}(\boldsymbol{s})$, where

$$\begin{aligned}
Z^{\mathrm{num}}_{V_3(\mathfrak{o})}(\boldsymbol{s}) &= 1 - t_2^2t_3^5 - qt_1t_2t_3^4 - t_1t_2^2t_3^5 - q^2t_1t_2^3t_3^3 - t_1t_2^3t_3^6 - qt_1t_2^3t_3^5 - q^2t_1t_2^3t_3^4 \\
&\quad + qt_1t_2^3t_3^6 - qt_1t_2^4t_3^5 + t_1t_2^3t_3^8 + qt_1t_2^3t_3^7 + q^2t_1t_2^3t_3^6 - q^2t_1^2t_2^3t_3^5 + t_1t_2^4t_3^8 \\
&\quad + qt_1t_2^4t_3^7 + q^2t_1^2t_2^3t_3^6 + qt_1^2t_2^3t_3^8 + q^2t_1t_2^5t_3^6 + qt_1t_2^5t_3^8 + q^2t_1^2t_2^4t_3^7 + q^3t_1^2t_2^4t_3^6 \\
&\quad - qt_1t_2^5t_3^9 + qt_1^2t_2^5t_3^8 + q^2t_1^2t_2^5t_3^7 + q^3t_1^2t_2^5t_3^6 - q^2t_1^2t_2^4t_3^9 + q^2t_1^2t_2^5t_3^8 \\
&\quad - qt_1^2t_2^5t_3^{10} - q^2t_1^2t_2^5t_3^9 - q^3t_1^2t_2^5t_3^8 - qt_1^2t_2^5t_3^{11} - q^3t_1^2t_2^6t_3^9 - q^2t_1^2t_2^7t_3^{10} \\
&\quad - q^3t_1^3t_2^6t_3^9 + q^3t_1^3t_2^8t_3^{14}, \\
Z^{\mathrm{den}}_{V_3(\mathfrak{o})}(\boldsymbol{s}) &= (1-t_2)(1-t_3)(1-qt_3)(1-t_3^3)(1-t_2^2t_3^3)(1-qt_2^2t_3^2)(1-q^2t_1t_2t_3) \\
&\quad \times (1-t_1t_2^2t_3^3)(1-qt_1t_2t_3^3)(1-q^2t_1t_2^2t_3^2)
\end{aligned}$$

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