Topological Manin pairs and (n, s)-type series

Raschid Abedin[*], Stepan Maximov[†] and Alexander Stolin[‡]

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Lie subalgebras of $L = \mathfrak{g}((x)) \times \mathfrak{g}[x]/x^n\mathfrak{g}[x]$, complementary to the diagonal embedding Δ of $\mathfrak{g}[\![x]\!]$ and Lagrangian with respect to some particular form, are in bijection with formal classical r-matrices and topological Lie bialgebra structures on the Lie algebra of formal power series $\mathfrak{g}[\![x]\!]$. In this work we consider arbitrary subspaces of L complementary to Δ and associate them with so-called series of type (n, s).

We prove that Lagrangian subspaces are in bijection with skew-symmetric (n, s)-type series and topological quasi-Lie bialgebra structures on $\mathfrak{g}[\![x]\!]$. Using the classificaiton of Manin pairs we classify up to twisting and coordinate transformations all quasi-Lie bialgebra structures.

Series of type (n, s), solving the generalized classical Yang-Baxter equation, correspond to subalgebras of L. We discuss their possible utility in the theory of integrable systems.

Dedicated to the memory of Yuri Manin

1 Introduction

Let F be an algebraically closed field of characteristic 0 equipped with the discrete topology and $\mathfrak{g}$ be a simple Lie algebra over F. We define the Lie algebra $\mathfrak{g}[\![x]\!]$ to be the space $\mathfrak{g} \otimes F[\![x]\!]$ with the bracket

$$[a \otimes f, b \otimes g] = [a, b] \otimes fg$$

and we equip it with the (x)-adic topology. The continuous dual of $\mathfrak{g}[\![x]\!]$ is denoted by $\mathfrak{g}[\![x]\!]'$ and it is endowed with the discrete topology.

A topological Manin pair is a pair $(L, \mathfrak{g}[\![x]\!])$ where

1. L is a Lie algebra equipped with an invariant non-degenerate symmetric bilinear form B;
2. $\mathfrak{g}[\![x]\!] \subset L$ is a Lagrangian subalgebra with respect to B;
3. for any continuous functional $T \colon \mathfrak{g}[\![x]\!] \to F$ there is $f \in L$ such that $T = B(f, -)$.

Topological Manin pairs were classified in [1] using the tools from [8]. More precisely, if $(L, \mathfrak{g}[\![x]\!])$ is a topological Manin pair, then L is isomorphic, as a Lie algebra with form, to either $L(\infty)$ or $L(n, \alpha)$. In this paper we consider only the "non-degenerate" case, namely $L \cong L(n, \alpha)$.

As a Lie algebra

$$L(n, \alpha) = \mathfrak{g}((x)) \times \mathfrak{g}[x]/x^n\mathfrak{g}[x].$$

The bilinear form B on $L(n, \alpha)$ is completely determined by the sequence $\alpha = (\alpha_i \in F \mid -\infty < i \leqslant n-2)$. For example, when $n = 0$ the form is given by

$$B(a \otimes f, b \otimes g) = \kappa(a, b)\mathrm{res}_0 \{\alpha(x)fg\},$$

where κ is the Killing form on $\mathfrak{g}$ and $\alpha(x) := 1 + \alpha_{-2}x + \alpha_{-3}x^2 + \cdots \in F((x))$. In case $n > 0$ the form is given by a similar formula; see Section 2.

It was established in [1], that the following objects are in one-to-one correspondence

[*]Department of Mathematics, ETH Zürich, 8006 Zürich, Schweiz, raschid.abedin@math.ethz.ch.
[†]Institut für Mathematik, Universität Paderborn, 33098 Paderborn, Germany, stmax@math.upb.de.
[‡]Department of Mathematical Sciences, Chalmers University of Technology and the University of Gothenburg, 412 96 Gothenburg, Sweden, astolin@chalmers.se

- Lagrangian subalgebras $W \subseteq L(n,0)$, $0 \leqslant n \leqslant 2$, complementary to the diagonal

$$\Delta := \{(f, [f]) \mid f \in \mathfrak{g}[\![x]\!]\},$$

  i.e. $\Delta \dotplus W = L(n,0)$;
- non-degenerate topological Lie bialgebra structures on $\mathfrak{g}[\![x]\!]$ and
- formal solutions to the classical Yang-Baxter equation (CYBE) in the form

$$\frac{y^n\Omega}{x-y} + g(x,y) = \Omega \sum_{k\geqslant 0} x^{-k-1}y^{k+n} + g(x,y) \in (\mathfrak{g}\otimes\mathfrak{g})(\!(x)\!)[\![y]\!], \tag{1}$$

  where $\Omega \in \mathfrak{g}\otimes\mathfrak{g}$ is the quadratic Casimir element and $g(x,y) \in (\mathfrak{g}\otimes\mathfrak{g})[\![x,y]\!]$.

Furthermore, the proof of the above-mentioned correspondence reveals that series Eq. (1) can be viewed as a generating series for the corresponding subalgebra W. The present paper can be thus considered as a continuation of [1], where we extend the preceding correspondence using series of type (n,s).

To define a series of type (n,s) fix a basis $\{b_i\}_{i=1}^d$ of $\mathfrak{g}$, orthonormal with respect to its Killing form κ, and interpret $y^n\Omega/(x-y)$ as a series

$$\frac{y^n\Omega}{x-y} = \sum_{k=0}^{\infty}\sum_{i=1}^{d} w_{k,i}\otimes b_i y^k \in \big((\mathfrak{g}(\!(x)\!)\times \mathfrak{g}[x]/x^n\mathfrak{g}[x])\otimes\mathfrak{g}\big)[\![y]\!]. \tag{2}$$

This expression might be understood as a Taylor series expansion. Elements $w_{k,i} \in \mathfrak{g}(\!(x)\!)\times\mathfrak{g}[x]/x^n\mathfrak{g}[x]$ are presented explicitly in Eq. (19). A series of type (n,s) is a series of the form

$$\frac{s(x)y^n\Omega}{x-y} + g(x,y) \in \big((\mathfrak{g}(\!(x)\!)\times \mathfrak{g}[x]/x^n\mathfrak{g}[x])\otimes\mathfrak{g}\big)[\![y]\!], \tag{3}$$

where $s \in F[\![x]\!]^\times$ and $g \in (\mathfrak{g}\otimes\mathfrak{g})[\![x,y]\!]$; See Definition 3.2. For each series r of type (n,s) we define another series $\overline{r}$ of the same type as follows

$$\overline{r} := \frac{s(y)x^n\Omega}{x-y} - \tau(g(y,x)), \tag{4}$$

where τ is the $F[\![x,y]\!]$-linear extension of the map $a\otimes b \mapsto b\otimes a$.

Each series of type (n,s) produces a subspace of $\mathfrak{g}(\!(x)\!)\times\mathfrak{g}[x]/x^n\mathfrak{g}[x]$ complementary to the diagonal embedding Δ of $\mathfrak{g}[\![x]\!]$. The following results generalize the above-mentioned correspondence from [1].

Theorem A. *Let $n\in\mathbb{Z}_{\geqslant 0}$ and $\alpha = (\alpha_i \in F \mid -\infty < i \leqslant n-2)$ be an arbitrary sequence with the corresponding series $\alpha(x) := x^{-n} + \alpha_{n-2}x^{-n+1} + \cdots + \alpha_0 x^{-1} + \cdots \in F(\!(x)\!)$. For any (n,s)-type series*

$$r = \sum_{k=0}^{\infty}\sum_{i=1}^{d} f_{k,i}\otimes b_i y^k \in \big((\mathfrak{g}(\!(x)\!)\times \mathfrak{g}[x]/x^n\mathfrak{g}[x])\otimes\mathfrak{g}\big)[\![y]\!] \tag{5}$$

define the space

$$W(r) := \operatorname{span}_F\{f_{k,i} \mid k\geqslant 0,\ 1\leqslant i\leqslant d\} \subseteq \mathfrak{g}(\!(x)\!)\times\mathfrak{g}[x]/x^n\mathfrak{g}[x]. \tag{6}$$

The following results are true:

1. *W defines a bijection between series of type $\left(n, \frac{1}{x^n\alpha(x)}\right)$ and subspaces $V \subset L(n,\alpha)$ complementary to the diagonal Δ, i.e. $L(n,\alpha) = \Delta \dotplus V$;*
2. *For any series r of type $\left(n, \frac{1}{x^n\alpha(x)}\right)$ we have $W(r)^\perp = W(\overline{r})$ inside $L(n,\alpha)$;*
3. *Any series r of type $\left(n, \frac{1}{x^n\alpha(x)}\right)$ satisfies $\mathrm{GCYB}(r) = \psi$ (see Definition 3.5 for the meaning of $\mathrm{GCYB}(r)$), where $\psi \in (\mathfrak{g}\otimes\mathfrak{g}\otimes\mathfrak{g})[\![x_1,x_2,x_3]\!]$ is defined by*

$$B(v_1\otimes v_2\otimes v_3, \psi) = B(v_1, [v_2, v_3])$$

   for all $v_1 \in W(\overline{r}), v_2, v_3 \in W(r)$.

In particular, considering the case when r is skew-symmetric, meaning $r = \overline{r}$, or when $\psi = 0$ we get the following correspondences.

Corollary B. *Let $n\in\mathbb{Z}_{\geqslant 0}$, $\alpha = (\alpha_i \in F \mid -\infty < i \leqslant n-2)$ and W be the map from Theorem A. Then*

1. *W defines a bijection between skew-symmtric $\left(n, \frac{1}{x^n\alpha(x)}\right)$-type series and Lagrangian subspaces $V \subseteq L(n,\alpha)$, complementary to the diagonal Δ;*
2. *W defines a bijection between $\left(n, \frac{1}{x^n\alpha(x)}\right)$-type series solving the GCYBE and subalgebras $V \subseteq L(n,\alpha)$ complementary to the diagonal Δ.*

Observe that an (n,s)-type series produces a subspace of $L(n,\alpha)$ for any sequence α. However, to obtain the compatibility with the form, given by α, we need the equality $s(x) = 1/(x^n\alpha(x))$. In this case, the components $f_{k,i}$ and $b_i y^k$ of the series become dual bases for $W(r)$ and Δ respectively.

The requirement on a series r of type (n,s) to solve the CYBE is equivalent to being skew-symmetric and to solve the GCYBE. Together with Corollary B this implies that Lagrangian subalgebras $W \subset L(n,\alpha)$, satisfying $W \dotplus \Delta = L(n,\alpha)$, are in bijection with $(n, 1/(x^n\alpha(x)))$-type series solving the classical Yang-Baxter equation. These correspondences are schematically depicted in Fig. 1.

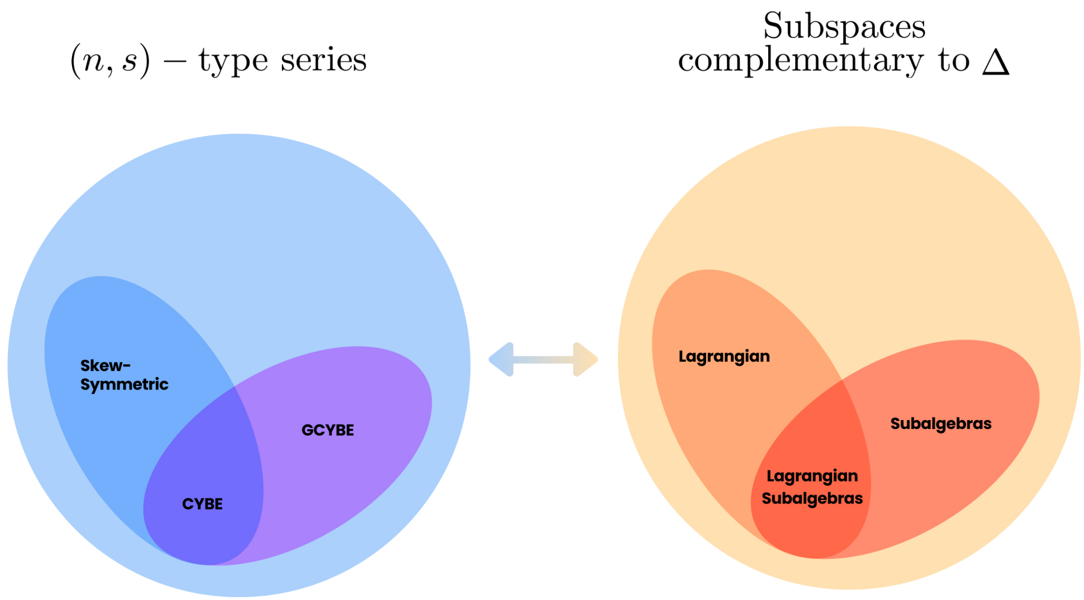

Figure 1: Series-subspaces correspondence

Remark 1.1. Let r be a series of type (n,s). Applying the projection $(a,b)\otimes c \mapsto a\otimes c$ onto the left component to r we obtain the series

$$r_{\text{proj}} = \frac{s(x)y^n\Omega}{x-y} + g(x,y) \in (\mathfrak{g}\otimes\mathfrak{g})((x))[\![y]\!]. \tag{7}$$

Conversely, starting with a series r_{proj} of the form Eq. (7), we can obtain an (n,s)-type series r by taking two Taylor series expansions of r_{proj} at $x = 0$ and $y = 0$ respectively and then constructing r by combining the coefficients of $b_i y^k$, $k \geqslant 0$, in these expansions. These two constructions are inverse to each other and hence both r and its projection r_{proj} contain exactly the same information. Consequently, all the statements made for (n,s)-type series can be stated in terms of their projections onto the left component and vice versa. In cotrast with [1], in this paper we give preference to series of type (n,s) rather than to their projections, because the statement that series of type (n,s) generate subspaces of $L(n,\alpha)$ becomes transparent. ◊

Reinterpreting the results of [1] in terms of (n,s)-type series we see that skew-symmetric series of type $(n, 1/(x^n\alpha(x))$, that also solve the GCYBE, exist only for $n = 0, 1$ and $n = 2$ with $\alpha_0 = 0$.

Lagrangian subalgebras of $L(n,\alpha)$, complementary to Δ, correspond to topological Lie bialgebra structures on $\mathfrak{g}[\![x]\!]$. If we instead consider Lagrangian subspaces (not necessarily subalgebras) of $L(n,\alpha)$, we get so called (non-degenerate) topological quasi-Lie bialgebra structures on $\mathfrak{g}[\![x]\!]$. A topological quasi-Lie bialgebra structure on $\mathfrak{g}[\![x]\!]$ consists of

- a skew-symmetric continuous linear map $\delta\colon \mathfrak{g}[\![x]\!] \to (\mathfrak{g}\otimes\mathfrak{g})[\![x,y]\!]$ and
- a skew-symmetric element $\varphi \in (\mathfrak{g}\otimes\mathfrak{g}\otimes\mathfrak{g})[\![x,y,z]\!]$,

which are subject to the following three conditions

1. $\delta([a,b]) = [a\otimes 1 + 1\otimes a, \delta(b)] - [b\otimes 1 + 1\otimes b, \delta(a)]$, i.e. δ is a 1-cocycle;

2. $\frac{1}{2}\mathrm{Alt}((\delta \otimes 1)\delta(a)) = [a \otimes 1 \otimes 1 + 1 \otimes a \otimes 1 + 1 \otimes 1 \otimes a, \varphi]$;

3. $\mathrm{Alt}((\delta \otimes 1 \otimes 1)\varphi) = 0$,

where $\mathrm{Alt}(x_1 \otimes \ldots \otimes x_n) := \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) x_{\sigma(1)} \otimes \ldots \otimes x_{\sigma(n)}$.

Following [5] we prove the following direct relation between δ, φ and skew-symmetric (n,s)-type series r.

Proposition C. *There is a bijection between topological quasi-Lie bialgebras and skew-symmetric (n,s)-type series. Let r be the (n,s)-type series corresponding to $(\mathfrak{g}[\![x]\!], \delta, \varphi)$, then, under the identification $\mathfrak{g}[\![x]\!] \cong \Delta$, we have the following identities:*

- $[a \otimes 1 + 1 \otimes a, r] = -\delta(a)$ *for any* $a \in \mathfrak{g}[\![x]\!]$ *and*
- $\mathrm{CYB}(r) = -\varphi$.

The same is true if r is interpreted as an element in $(\mathfrak{g} \otimes \mathfrak{g})(\!(x)\!)[\![y]\!]$.

In view of this result we call skew-symmetric (n,s)-type series quasi-r-matrices.

Repeating the ideas from [7] and [5] we show that topological quasi-Lie bialgebras can be twisted similar to topological Lie bialgebras. More precisely, if δ is a quasi-Lie bialgebra structure on $\mathfrak{g}[\![x]\!]$, given by the Lagrangian subspace W, and $s := \sum_i a_i \otimes b^i \in (\mathfrak{g} \otimes \mathfrak{g})[\![x,y]\!]$ is an arbitrary skew-symmetric tensor, then

$$W_s := \left\{ \sum_i B(b^i, w) a_i - w \mid w \in W \right\} \tag{8}$$

is another (twisted) Lagrangian subspace complementary to the diagonal. This observation implies, that in order to classify all topological quasi-Lie bialgebra structures on $\mathfrak{g}[\![x]\!]$ up to twisting it is enough to find one single Lagrangian subspace within each $L(n,\alpha)$. Moreover, it was shown in [1] that substitutions of the form $x \mapsto x + a_2 x^2 + a_3 x^3 + \ldots$, $a_i \in F$, allow us to assume that α has the form

$$\alpha = (\ldots, 0, \alpha_0, 0, \ldots, 0).$$

Lagrangian subspaces for such $L(n,\alpha)$ are constructed in Section 4.1.

Using Theorem A and Proposition C we explain how twisting of a Lagrangian subspace $W \subset L(n,\alpha)$ is seen at the level of δ and the corresponding quasi-r-matrix r.

Corollary D. *Let $(\mathfrak{g}[\![x]\!], \delta, \varphi)$ be a topological quasi-Lie bialgebra structure corresponding to the quasi-r-matrix r. If we twist $W(r)$ with a skew-symmetric tensor s we obtain another topological quasi-Lie bialgebra $(\mathfrak{g}[\![x]\!], \delta_s, \varphi_s)$, such that*

1. $W(r)_s = W(r - s)$;
2. $\delta_s = \delta + ds$;
3. $\varphi_s = \varphi + \mathrm{CYB}(s) - \frac{1}{2}\mathrm{Alt}((\delta \otimes 1)s)$.

Therefore, to describe all quasi-r-matrices up to twisting it is enough to find one single quasi-r-matrix for each $L(n,\alpha)$. We achieve that goal in Section 4.2 by writing out explicitly series of type (n,s) for subspaces from Section 4.1.

The results above, in particular, show that if r is a quasi-r-matrix and $\delta(a) := [a \otimes 1 + 1 \otimes a, r]$, then the condition

$$\mathrm{Alt}((\delta \otimes 1 \otimes 1)\mathrm{CYB}(r)) = 0 \tag{9}$$

is trivially satisfied.

We conclude the paper by using Theorem A for construction of Lie algebra splittings $\Delta \dotplus W = L(n,\alpha)$ and the corresponding (n,s)-type series, which we call generalized r-matrices. These constructions are important in the theory of integrable systems because of their use in the Adler-Konstant-Symes (AKS) scheme and the so-called r-matrix approach; see [4, 6]. The subalgebra splittings of $L(0,0)$ as well as their physical applications were considered in e.g. [9, 10].

Our first result shows that in order to obtain new generalized r-matrices from subalgebra splittings $L(n,\alpha) = \Delta \dotplus W$ with $n > 2$, the subalgebra W must be unbounded. Otherwise the situation can be reduced to the splitting of $L(2,\alpha)$.

Proposition E. *Let $L(n,\alpha) = \Delta \dotplus W$ for some subalgebra $W \subset L(n,\alpha)$ and $n > 2$. Assume W is bounded, i.e. there is an integer $N > 0$ such that*

$$x^{-N}\mathfrak{g}[x^{-1}] \subseteq W_+ \subseteq x^N \mathfrak{g}[x^{-1}],$$

where W_+ is the projection of $W \subset L(n,\alpha) = \mathfrak{g}((x)) \oplus \mathfrak{g}[x]/x^n\mathfrak{g}[x]$ on the first component $\mathfrak{g}((x))$. Then we have the inclusion

$$\{0\} \times [x^2]\mathfrak{g}[x]/x^n\mathfrak{g}[x] \subseteq W$$

and the image $\widetilde{W}$ under the canonical projection $L(n,\alpha) \to L(2,\alpha)$ is a subalgebra satisfying $L(2,\alpha) = \Delta \dotplus \widetilde{W}$.

Despite this result we think that bounded subalgebras $W \subset L(n,\alpha)$ complementary to Δ are still interesting, because in the case $\alpha \neq 0$ they lead to unbounded orthogonal complements $W^\perp$ which are also important in view of the AKS scheme. We give examples of subalgebras of $L(n,\alpha)$ with unbounded orthogonal complements.

2 Topological Manin pairs

Let F be an algebraically closed field of characteristic 0, $\mathfrak{g}$ be a finite-dimensional simple F-Lie algebra and $\mathfrak{g}[\![x]\!] := \mathfrak{g} \otimes F[\![x]\!]$ be the Lie algebra with the bracket defined by

$$[a \otimes f, b \otimes g] := [a,b] \otimes fg,$$

for all $a, b \in \mathfrak{g}$ and $f, g \in F[\![x]\!]$. From now on, we always endow F with the discrete topology and view $\mathfrak{g}[\![x]\!]$ as a topological Lie algebra with the (x)-adic topology.

A *topological Manin pair* is a pair $(L, \mathfrak{g}[\![x]\!])$, where L is a Lie algebra equipped with an invariant non-degenerate symmetric bilinear form B, such that

1. $\mathfrak{g}[\![x]\!] \subseteq L$ is a Lagrangian Lie subalgebra with respect to B;
2. for any continuous functional $T\colon \mathfrak{g}[\![x]\!] \to F$ there exists an element $f \in L$ such that $T = B(f, -)$.

The statements of [8, Proposition 2.9] and [1, Proposition 3.12] give a description of all topological Manin pairs. For precise formulation we need to repeat the definitions of some specific Lie algebras with forms from [1, Section 3.2] and [8, Section 2].

Definition 2.1. We define the Lie algebra $L(\infty) := \mathfrak{g} \otimes A(\infty)$, where $A(\infty)$ is the unital commutative algebra with underlying space $\sum_{i \geqslant 0} F a_i \dotplus F[\![x]\!]$ and multiplication given by

$$a_i a_j := 0,\ a_i x^j := a_{i-j} \text{ for } i \geqslant j \text{ and } a_i x^j := 0 \text{ otherwise.}$$

Let $\mathrm{t}\colon A \to F$ be the functional, given by $\mathrm{t}(a_0) := 1$, $\mathrm{t}(a_i) := 0$, $i \geqslant 1$ and $\mathrm{t}(F[\![x]\!]) := 0$. We equip $L(\infty)$ with the symmetric non-degenerate invariant bilinear form

$$B\left(a \otimes \left(\sum_{i \geqslant 0} c_i a_i, f(x)\right), b \otimes \left(\sum_{i \geqslant 0} t_i a_i, g(x)\right)\right) := \kappa(a,b)\, \mathrm{t}\left(g(x) \sum_{i \geqslant 0} c_i a_i + f(x) \sum_{i \geqslant 0} t_i a_i\right). \tag{10}$$

◊

Definition 2.2. Let $n \geqslant 1$ and $\alpha = (\alpha_i \in F \mid -\infty < i \leqslant n-2)$ be an arbitrary sequence. Consider the algebra

$$A(n,\alpha) := F((x)) \oplus F[x]/(x^n).$$

Abusing the notation we denote the element $x^{-n} + \alpha_{n-2}x^{-n+1} + \dots + \alpha_0 x^{-1} + \dots \in F((x))$ with the same letter α. Define the functional $\mathrm{t}\colon A(n,\alpha) \to F$ by

$$\mathrm{t}(f, [p]) := \mathrm{res}_0 \{\alpha(f - p)\}.$$

Taking the tensor product of $A(n,\alpha)$ with $\mathfrak{g}$ we get the Lie algebra $L(n,\alpha) := \mathfrak{g} \otimes A(n,\alpha)$, which we equip with the form

$$B(a \otimes (f, [p]), b \otimes (g, [q])) := \kappa(a,b)\, \mathrm{t}(fg, [pq]). \tag{11}$$

It is known that the bilinear form B is symmetric non-degenerate and invariant. ◊

Definition 2.3. Take an arbitrary sequence $\alpha = (\alpha_i \in F \mid -\infty < i \leqslant -2)$ and let $A(0,\alpha) := F((x))$. We define the functional $\mathrm{t}\colon A(0,\alpha) \to F$ by

$$\mathrm{t}(f) := \mathrm{res}_0 \{\alpha f\},$$

where $\alpha = 1 + \alpha_{-2}x + \dots \in F((x))$. We equip the Lie algebra $L(0,\alpha) := \mathfrak{g} \otimes A(0,\alpha)$ with the bilinear form

$$B(a \otimes f, b \otimes g) := \kappa(a,b)\, \mathrm{t}(fg), \tag{12}$$

which is again symmetric non-degenerate and invariant. From now on we identify $F((x))$ with $F((x)) \times \{0\}$ and write $(f, 0)$ for elements in $A(0,\alpha)$. ◊

Definition 2.4. A series of the form $\varphi = x + a_2x^2 + a_3x^3 + \cdots \in F[\![x]\!]$ is called a *coordinate transformation*. Coordinate transformations form a group $\mathrm{Aut}_0F[\![x]\!]$ under substitution which we view as a subgroup of automorphisms of $F[\![x]\!]$.

An element $\varphi \in \mathrm{Aut}_0F[\![x]\!]$ induces an automorphism of $A(n,\alpha)$ by $f/g \mapsto \varphi(f)/\varphi(g)$ and $[p] \mapsto [\varphi(p)]$ that changes the functional t to $\mathrm{t}\circ\varphi$. We write $A(n,\alpha)^{(\varphi)}$ for the algebra $A(n,\alpha)$ with the functional $\mathrm{t}\circ\varphi$. It is not hard to see that for any $\varphi \in \mathrm{Aut}_0F[\![x]\!]$ there is a sequence β such that $A(n,\alpha)^{(\varphi)} = A(n,\beta)$. ♢

Let $(L, \mathfrak{g}[\![x]\!])$ be a topological Manin pair. According to [8, Proposition 2.9] as a Lie algebra with form $L \cong L(\infty)$ or $L \cong L(n,\alpha)$, for some $n \geqslant 0$ and some sequence α. Here we identify $\mathfrak{g}[\![x]\!]$ with the diagonal

$$\Delta \coloneqq \{(f,[f]) \mid f \in \mathfrak{g}[\![x]\!]\} \subset L(n,\alpha).$$

Moreover, we can assume that all the elements α_i in the sequence α, except maybe α_0, are 0 by virtue of the following result.

Proposition 2.5. *[1, Proposition 3.12] Let $n \geqslant 0$ and $\alpha = (\alpha_i \in F \mid -\infty < i \leqslant n-2)$ be a sequence. There exists a $\varphi \in \mathrm{Aut}_0F[\![x]\!]$ such that $A(n,\alpha) \cong A(n,\beta)^{(\varphi)}$, where β is the sequence satisfying $\beta_i = 0$ for all $i \neq 0$ and $\beta_0 = \alpha_0$.*

Remark 2.6. Observe that the result of Proposition 2.5 can be interpreted in terms of a formal differential equation. Consider an arbitrary $\alpha(x) = x^{-n} + \alpha_{n-2}x^{-n+1} + \cdots + \alpha_0x^{-1} + \cdots \in F(\!(x)\!)$ and $\beta(x) = x^{-n} + \alpha_0x^{-1}$. Then the functionals t_α and t_β defined on $A(n,\alpha)$ and $A(n,\beta)$ respectively are given by

$$\mathrm{t}_\alpha(f,[p]) = \mathrm{res}_0\{\alpha(f-p)\} \quad \text{and} \quad \mathrm{t}_\beta(f,[p]) = \mathrm{res}_0\{\beta(f-p)\})$$

The equality $A(n,\alpha)^{(\varphi)} = A(n,\beta)$ can be expressed as

$$\mathrm{res}_0\{\beta(x)f(x)\} = \mathrm{res}_0\{\alpha(x)f(\varphi(x))\} = \mathrm{res}_0\{\alpha(\psi(x))f(x)\psi'(x)\}, \tag{13}$$

where $\psi \in \mathrm{Aut}_0(F[\![x]\!])$ is the compositional inverse of φ, i.e. $\varphi(\psi(x)) = x$. Since the residue pairing is non-degenerate on $F(\!(x)\!)$, we obtain

$$\alpha(\psi(x))\psi'(x) = \beta(x). \tag{14}$$

In particular, the transformation φ is the compositional inverse of the solution to Eq. (14). ♢

3 Series of type (n, s) and subspaces of $L(n, \alpha)$

Let $\{b_i\}_{i=1}^d$ be an othonormal basis of $\mathfrak{g}$ with respect to the Killing form κ. We write Ω for the quadratic Casimir element $\sum_{i=1}^d b_i \otimes b_i \in \mathfrak{g}\otimes\mathfrak{g}$. It satisfies the identity $[a\otimes 1 + 1\otimes a, \Omega] = 0$ for all $a \in \mathfrak{g}$.

In this section we describe a bijection between subspaces $W \subset L(n,\alpha)$ complementary to Δ and certain series. The following definition introduces convenient spaces containing these series.

Definition 3.1. We put $A_1(n,\alpha) \coloneqq A(n,\alpha) = F(\!(x_1)\!) \oplus F[x_1]/(x_1^n)$ and then define inductively the algebras

$$A_m(n,\alpha) \coloneqq A_{m-1}(n,\alpha)(\!(x_m)\!) \oplus A_{m-1}(n,\alpha)[x_m]/x_m^nA_{m-1}(n,\alpha), \;\; m > 1. \tag{15}$$

The functional t defined on $A(n,\alpha)$ extends inductively to a functional on $A_m(n,\alpha)$. More precisely,

$$\mathrm{t}\left(\sum_{k\geqslant -N} f_kx_m^k, \sum_{\ell=0}^{n-1}[g_\ell x_m^\ell]\right) \coloneqq \sum_{k\geqslant -N}\mathrm{t}(f_k)\,\mathrm{t}(x_m^k,0) + \sum_{\ell=0}^{n-1}\mathrm{t}(g_\ell)\,\mathrm{t}(0,[x_m]^\ell), \tag{16}$$

where $f_k, g_\ell \in A_{m-1}(n,\alpha)$. Since $\mathrm{t}(x^nF[\![x]\!]) = 0$, the sum on the right-hand side of Eq. (16) is finite and well-defined. This allows us to extend the form B on $L(n,\alpha)$ to a symmetric non-degenerate bilinear form on the $\mathfrak{g}$-module

$$L_m(n,\alpha) \coloneqq \mathfrak{g}^{\otimes m} \otimes A_m(n,\alpha) \tag{17}$$

by letting

$$B((a_1\otimes\ldots\otimes a_m)\otimes f, (b_1\otimes\ldots\otimes b_m)\otimes g) \coloneqq \mathrm{t}(fg)\prod_{k=1}^m \kappa(a_k,b_k), \tag{18}$$

for all $a_1,\ldots,a_m,b_1,\ldots,b_m \in \mathfrak{g}$ and $f,g \in A_m(n,\alpha)$. ♢

Fix some integer $n \geqslant 0$. We interpret the quotient $y^n\Omega/(x-y)$ in the following way

$$\begin{aligned}\frac{y^n\Omega}{x-y} &= \sum_{k=0}^{n-1}\sum_{i=1}^{d} b_i(0,-[x]^{(n-1)-k})\otimes b_i(y^k,[y]^k) + \sum_{k=n}^{\infty}\sum_{i=1}^{d} b_i(x^{(n-1)-k},0)\otimes b_i(y^k,0) \\ &= \sum_{k=0}^{\infty}\sum_{i=1}^{d} w_{k,i}\otimes b_i(y^k,[y]^k) \in (L(n,\alpha)\otimes\mathfrak{g})\,[\![(y,[y])]\!] \subset L_2(n,\alpha),\end{aligned} \tag{19}$$

where α is an arbitrary sequence and we write $b_i(x^\ell,[x]^m)$ meaning $b_i\otimes(x^\ell,[x]^m)$.

Definition 3.2. Since $(L(n,\alpha)\otimes\mathfrak{g})\,[\![(y,[y])]\!]$ is an $F[\![x]\!]\cong F[\![(x,[x])]\!]$-module and

$$(\mathfrak{g}\otimes\mathfrak{g})[\![x,y]\!]\cong(\Delta\otimes\mathfrak{g})[\![(y,[y])]\!]\subset(L(n,\alpha)\otimes\mathfrak{g})\,[\![(y,[y])]\!]$$

the series

$$r(x,y)=\frac{s(x)y^n\Omega}{x-y}+g(x,y), \tag{20}$$

where $g\in(\mathfrak{g}\otimes\mathfrak{g})[\![x,y]\!]$ and $s\in F[\![x]\!]^\times$, is also inside $(L(n,\alpha)\otimes\mathfrak{g})\,[\![(y,[y])]\!]$. Series of the form Eq. (20) are called *series of type* (n,s). ♢

Remark 3.3. Every series

$$r(x,y)=\frac{h(x,y)\Omega}{x-y}+g(x,y)\in L_2(n,\alpha),$$

where $h\in F[\![x,y]\!]$, $h(x,x)\neq 0$ and $g\in(\mathfrak{g}\otimes\mathfrak{g})[\![x,y]\!]$, has a unique representation as a series of type (n,s). Indeed, write $h(x,x)=x^ns(x)$ for some $s\in F[\![x]\!]^\times$. Then $h(x,y)-y^ns(x)=(x-y)f(x,y)$ for some $f\in F[\![x,y]\!]$. This implies that we can rewrite r in the (n,s) form

$$r(x,y)=\frac{s(x)y^n\Omega}{x-y}+f(x,y)\Omega+g(x,y). \tag{21}$$

In the construction of f we are using the fact that for any F-vector space V and any element $h\in V[\![x,y]\!]$

$$h(z,z)=0\implies h(x,y)=(x-y)f(x,y) \tag{22}$$

for some $f\in V[\![x,y]\!]$. ♢

Definition 3.4. For each series r of type (n,s) we define another series $\overline{r}$ of the same type (n,s) by

$$\overline{r}(x,y):=\frac{s(y)x^n\Omega}{x-y}-\tau(g(y,x))\in(L(n,\alpha)\otimes\mathfrak{g})\,[\![(y,[y])]\!], \tag{23}$$

where τ is the $F[\![x,y]\!]$-linear extension of the map $a\otimes b\mapsto b\otimes a$. To see that this is an (n,s)-type series its enough to apply the argument from Remark 3.3. Series of type (n,s), satisfying $r=\overline{r}$, are called *skew-symmetric.* ♢

Definition 3.5. *The generalized classical Yang-Baxter equation (GCYBE)* is the equation for an (n,s)-type series of the form

$$\mathrm{GCYB}(r):=[r^{12}(x_1,x_2),r^{13}(x_1,x_3)]+[r^{12}(x_1,x_2),r^{23}(x_2,x_3)]+[r^{13}(x_1,x_3),\overline{r}^{23}(x_2,x_3)]=0. \tag{24}$$

Here $(-)^{13}\colon L_2(n,\alpha)\to(U(\mathfrak{g})\otimes U(\mathfrak{g})\otimes U(\mathfrak{g}))\otimes A_3(n,\alpha)$ is the inclusion map given by

$$a\otimes b\otimes\left(\sum_{k\geqslant -N}F(x_1,[x_1])x_2^k,\sum_{m=0}^{n-1}G(x_1,[x_1])[x_2]^m\right)\mapsto a\otimes 1\otimes b\otimes\left(\sum_{k\geqslant -N}F(x_1,[x_1])x_3^k,\sum_{m=0}^{n-1}G(x_1,[x_1])[x_3]^m\right).$$

Other inclusions are defined in a similar manner. The commutators are then taken in the associative $A_3(n,\alpha)$-algebra $(U(\mathfrak{g})\otimes U(\mathfrak{g})\otimes U(\mathfrak{g}))\otimes A_3(n,\alpha)$. ♢

Before formulating the main theorem of the section we note that if $\alpha=(\alpha_i\in F\mid -\infty<i\leqslant n-2)$ is an arbitrary sequence and $\alpha(x)=x^{-n}+\alpha_{n-2}x^{-n+1}+\cdots+\alpha_0x^{-1}+\cdots\in F(\!(x)\!)$ is the corresponding series, then $x^n\alpha(x)\in F[\![x]\!]^\times$.

Theorem 3.6. *Let $n \in \mathbb{Z}_{\geqslant 0}$ and $\alpha = (\alpha_i \in F \mid -\infty < i \leqslant n-2)$ be an arbitrary sequence with the corresponding series $\alpha(x) \in F(\!(x)\!)$. Consider the map*

$$W\colon L_2(n,\alpha) \longrightarrow \{V \subset L(n,\alpha) \mid V \text{ is a subspace}\}$$

given by

$$\sum_{i,j} b_i \otimes b_j \otimes \left(\sum_{k \geqslant -N_i} (f_k^{ij}, [p_k^{ij}])x^k, \sum_{m=0}^{n-1} (g_m^{ij}, [q_m^{ij}])[x]^m \right) \mapsto \mathrm{span}_F \left\{ b_i(f_k^{ij}, [p_k^{ij}]) \mid k \geqslant -N, 1 \leqslant i,j \leqslant d \right\}.$$

The following results are true:

1. *W defines a bijection between series of type $\left(n, \frac{1}{x^n\alpha(x)}\right)$ and subspaces $V \subseteq L(n,\alpha)$ complementary to the diagonal Δ, i.e. $L(n,\alpha) = \Delta \dotplus V$;*
2. *For any series r of type $\left(n, \frac{1}{x^n\alpha(x)}\right)$ we have $W(r)^\perp = W(\overline{r})$ inside $L(n,\alpha)$;*
3. *Any series r of type $\left(n, \frac{1}{x^n\alpha(x)}\right)$ satisfies $\mathrm{GCYB}(r) = \psi$, where $\psi \in (\mathfrak{g}\otimes\mathfrak{g}\otimes\mathfrak{g})[\![(x_1,[x_1]),(x_2,[x_2]),(x_3,[x_3])]\!]$ is defined by*
$$B(v_1 \otimes v_2 \otimes v_3, \psi) = B(v_1, [v_2, v_3])$$
for all $v_1 \in W(\overline{r}), v_2, v_3 \in W(r)$.

Proof. Fix an $\left(n, \frac{1}{x^n\alpha(x)}\right)$-type series

$$\begin{aligned} r(x,y) &= \frac{1}{x^n\alpha(x)} \frac{y^n\Omega}{x-y} + g(x,y) \\ &= \sum_{k=0}^{\infty}\sum_{i=1}^{d} s_{k,i} \otimes b_i(y^k,[y]^k) + \sum_{k=0}^{\infty}\sum_{i=1}^{d} g_{k,i} \otimes b_i(y^k,[y]^k) \in (L(n,\alpha)\otimes\mathfrak{g})[\![(y,[y])]\!]. \end{aligned}$$

It is easy to see that

$$U := \mathrm{span}_F\{w_{k,i} \mid k \geqslant 0, 1 \leqslant k \leqslant d\} \subset L(n,\alpha),$$

where $w_{k,i}$ are defined in Eq. (19), satisfies the condition $\Delta \dotplus U = L(n,\alpha)$. Since $s := \frac{1}{x^n\alpha(x)}$ is invertible, we have $sU \dotplus s\Delta = sU \dotplus \Delta = L(n,\alpha)$. In other words, the space

$$sU = \mathrm{span}_F\{s_{k,i} = sw_{k,i} \mid k \geqslant 0, 1 \leqslant k \leqslant d\} \subset L(n,\alpha) \tag{25}$$

is also complementary to the diagonal. Finally, since $g_{k,i} \in \Delta$ the space

$$W(r) = \mathrm{span}_F\{sw_{k,i} + g_{k,i} \mid k \geqslant 0, 1 \leqslant k \leqslant d\} \subset L(n,\alpha)$$

is complementary to the diagonal. Conversely, if $V \subset L(n,\alpha)$ satisfies $V \dotplus \Delta = L(n,\alpha)$, then for each $k \geqslant 0$ and $1 \leqslant i \leqslant d$ we can find a unique $g_{k,i} \in \Delta$ such that $sw_{k,i} + g_{k,i} \in V$. Define the (n,s) series r_V by

$$r_V(x,y) = \sum_{k\geqslant 0}\sum_{i=1}^{d} (sw_{k,i} + g_{k,i}) \otimes b_i(y^k,[y]^k).$$

It is now clear, that $W(r_V) = V$. These constructions establish the bijection in part 1.

To prove the second statement, observe that

$$B(sw_{k,i}, b_j(y^\ell,[y]^\ell)) = \delta_{i,j}\delta_{k,\ell}. \tag{26}$$

Furthermore, the straightforward calculation shows that

$$\begin{aligned} B(sw_{k,i}, sw_{\ell,j}) &= \begin{cases} -\mathrm{res}_0\left\{sx^{(n-1)-k-\ell-1}\right\} & \text{if } i=j \text{ and } 0 \leqslant k,\ell \leqslant n-1, \\ \mathrm{res}_0\left\{sx^{(n-1)-k-\ell-1}\right\} & \text{if } i=j \text{ and } k,\ell \geqslant n, \\ 0 & \text{otherwise,} \end{cases} \\ &= \begin{cases} -s_{k+\ell-n+1} & \text{if } i=j,\ 0 \leqslant k,\ell \leqslant n-1 \text{ and } k+\ell \geqslant n-1, \\ s_{k+\ell-n+1} & \text{if } i=j \text{ and } k,\ell \geqslant n, \\ 0 & \text{otherwise,} \end{cases} \end{aligned}$$

where $s(x) = \sum_{k=0}^{\infty} s_k x^k$. We write

$$\begin{aligned}\overline{r}(x,y) &= \frac{s(y)x^n\Omega}{x-y} - \tau(g(y,x)) = \frac{s(x)y^n\Omega}{x-y} - \frac{(s(x)y^n - s(y)x^n)\Omega}{x-y} - \tau(g(y,x)) \\ &= \sum_{k\geqslant 0}\sum_{i=1}^{d}(sw_{k,i} + \overline{g}_{k,i}) \otimes b_i(y^k, [y]^k).\end{aligned}$$

Consider the quotient

$$\begin{aligned}&\frac{(s(x)y^n - s(y)x^n)\Omega}{x-y} = \frac{y^n(s(x)-s(y))\Omega}{x-y} - \frac{s(y)(x^n - y^n)\Omega}{x-y} \\ &= \sum_{k\geqslant 0}\sum_{i=1}^{d} s_k \left(\sum_{\ell=1}^{k} b_i(x^{k-\ell},[x]^{k-\ell}) \otimes b_i(y^{(n-1)+\ell},[y]^{(n-1)+\ell}) - \sum_{\ell=1}^{n} b_i(x^{n-\ell},[x]^{n-\ell}) \otimes b_i(y^{k+\ell-1},[y]^{k+\ell-1})\right).\end{aligned}$$

The coefficient of $b_i(x^k,[x]^k) \otimes b_i(y^\ell,[y]^\ell)$ in the expression above is

$$\begin{aligned}-s_{k+\ell-(n-1)} &\quad \text{if } 0 \leqslant k,\ell \leqslant n-1 \text{ and } k+\ell \geqslant n-1, \\ s_{k+\ell-(n-1)} &\quad \text{if } k,\ell \geqslant n,\end{aligned}$$

which coincides with $B(sw_{k,i}, sw_{\ell,i})$. If we now expand the coefficients $g_{k,i}$ in the following way

$$g_{k,i} = \sum_{\ell\geqslant 0}\sum_{j=1}^{d} g_{k,i}^{\ell,j} b_j(x^\ell,[x]^\ell),$$

the coefficients $\overline{g}_{k,i}$ can be rewritten as

$$\overline{g}_{k,i} = -\sum_{\ell\geqslant 0}\sum_{j=1}^{d}(g_{\ell,j}^{k,i} + B(sw_{k,i}, sw_{\ell,j}))b_i(x^k,[x]^k) \otimes b_j(y^\ell,[y]^\ell).$$

Combining all the results above we obtain the desired equality

$$\begin{aligned}B(sw_{k,i} + g_{k,i}, sw_{\ell,j} + \overline{g}_{\ell,j}) &= B(sw_{k,i}, sw_{\ell,j}) + B(sw_{k,i}, \overline{g}_{\ell,j}) + B(g_{k,i}, sw_{\ell,j}) + B(g_{k,i}, \overline{g}_{\ell,j}) \\ &= B(sw_{k,i}, sw_{\ell,j}) + (-g_{k,i}^{\ell,j} - B(sw_{k,i}, sw_{\ell,j})) + g_{k,i}^{\ell,j} + 0 \\ &= 0\end{aligned}$$

which completes the proof of the second statement.

Using the same technique as in [2, Section 1], one can prove that

$$\psi := \mathrm{GCYB}(r) \in (\Delta \otimes \mathfrak{g} \otimes \mathfrak{g})[\![(x_2,[x_2]),(x_3,[x_3])]\!]$$

for any series r of type (n,s). Define $r_{k,i} := sw_{k,i} + g_{k,i}$ and $\overline{r}_{k,i} := sw_{k,i} + \overline{g}_{k,i}$ and rewrite $\mathrm{GCYB}(r)$ as

$$\begin{aligned}\psi &= \sum_{k,\ell\geqslant 0}\sum_{i,j=1}^{d}[r_{k,i}, r_{\ell,j}] \otimes b_i(x_2^k,[x_2]^k) \otimes b_j(x_3^\ell,[x_3]^\ell) \\ &\quad + \sum_{k\geqslant 0}\sum_{i=1}^{d} r_{k,i} \otimes \left([b_i(x_2^k,[x_2]^k) \otimes (1,1), r(x_2,x_3)] + [(1,1) \otimes b_i(x_3^k,[x_3]^k), \overline{r}(x_2,x_3)]\right).\end{aligned} \tag{27}$$

Applying $B(\overline{r}_{k_1,i_1} \otimes r_{k_2,i_2} \otimes r_{k_3,i_3}, -)$ to the equation above, we get

$$B(\overline{r}_{k_1,i_1} \otimes r_{k_2,i_2} \otimes r_{k_3,i_3}, \psi) = B(\overline{r}_{k_1,i_1}, [r_{k_2,i_2}, r_{k_3,i_3}]). \tag{28}$$

This gives the last statement because $W(r)$ and $W(\overline{r})$ are generated by $r_{k,i}$ and $\overline{r}_{k,i}$ respectively. ■

Corollary 3.7. *Let $n \in \mathbb{Z}_{\geqslant 0}$, $\alpha = (\alpha_i \in F \mid -\infty < i \leqslant n-2)$ and W be as in Theorem 3.6. Then*

1. *W defines a bijection between skew-symmtric $\left(n, \frac{1}{x^n\alpha(x)}\right)$-type series and Lagrangian subspaces $V \subseteq L(n,\alpha)$ complementary to the diagonal Δ;*
2. *W defines a bijection between $\left(n, \frac{1}{x^n\alpha(x)}\right)$-type series solving GCYBE and subalgebras $V \subseteq L(n,\alpha)$ complementary to the diagonal Δ.*

As we can see from the proof of Theorem 3.6 the element ψ in $\mathrm{GCYB}(r) = \psi$ represents the obstruction for $W(r)$ from being a Lie subalgebra. This observation raises an interesting question that we do not consider in this paper: what elements ψ can appear on the right-hand side of the above-mentioned equation.

Observe that if r is a series of type (n,s) and it satisfies

$$\mathrm{CYB}(r) := [r^{12}(x_1,x_2), r^{13}(x_1,x_3)] + [r^{12}(x_1,x_2), r^{23}(x_2,x_3)] + [r^{13}(x_1,x_3), r^{23}(x_2,x_3)] = \psi \tag{29}$$

for some $\psi \in (\mathfrak{g}\otimes\mathfrak{g}\otimes\mathfrak{g})[\![x,y,z]\!]$, then r is automatically skew-symmetric and hence solves the first equation as well. To prove that one can e.g. repeat the argument from [1, Lemma 5.2]. In other words, for a fixed ψ solutions to $\mathrm{CYB}(r) = \psi$ form a subclass of solutions to $\mathrm{GCYB}(r) = \psi$. In particular, solutions to $\mathrm{CYB}(r) = 0.$ are exactly the skew-symmetric solutions to $\mathrm{GCYB}(r) = 0$. We call the equation $\mathrm{CYB}(r) = \psi$ *Manin-Yang-Baxter equation.*

Remark 3.8. As our notation suggest, we could have interpreted $y^n\Omega/(x-y)$ as

$$\frac{y^n\Omega}{x-y} = \sum_{k\geqslant 0}\sum_{i=1}^{d} b_i x^{-k-1}\otimes b_i y^{n+k} \in (\mathfrak{g}\otimes\mathfrak{g})(\!(x)\!)[\![y]\!]$$

and performed all the arithmetic calculations in this form. To restore an (n,s)-type series from

$$\frac{s(x)y^n\Omega}{x-y} + g(x,y) \in (\mathfrak{g}\otimes\mathfrak{g})(\!(x)\!)[\![y]\!] \tag{30}$$

we can simply view $s(x)\in F[\![x]\!]^\times$ and $g(x,y)\in(\mathfrak{g}\otimes\mathfrak{g})[\![x,y]\!]$ as elements in $F[\![(x,[x])]\!]^\times$ and $(\mathfrak{g}\otimes\mathfrak{g})[\![(x,[x]),(y,[y])]\!]$ respectively and reinterpret the singular part $y^n\Omega/(x-y)$ as it was done in Eq. (19).

Conversely, to get a series of the form Eq. (30) from a series of type (n,s) we can just project the latter onto the first component.

In other words, we have a bijection between (n,s)-type series in $L_2(n,\alpha)$ and their projections Eq. (30) onto the first component given by different interpretations of the singular part $y^n\Omega/(x-y)$.

Although, all arithmetic operations can be performed in the form Eq. (30), the construction of $W(r)$ and statements like $\Delta\cap W(r) = 0$ require us to pass to the interpretation Eq. (19). This is our main motivation to work directly with (n,s)-type series in $L_2(n,\alpha)$ instead of their projections. $\Diamond$

In view of Remark 3.8, we have a new proof of [1, Corollary 5.5].

Corollary 3.9. *Classical (formal) r-matrices, i.e. skew-symmetric elements*

$$\frac{s(x)y^n\Omega}{x-y} + g(x,y) = \frac{1}{x^n\alpha(x)}\frac{y^n\Omega}{x-y} + g(x,y) \in (\mathfrak{g}\otimes\mathfrak{g})(\!(x)\!)[\![y]\!], \tag{31}$$

solving GCYBE, are in bijection with skew-symmetric series of type (n,s) solving GCYBE and hence in bijection with Lagrangian Lie subalgebras of $L(n,\alpha)$ complementary to the diagonal Δ.

The result of [1, Theorem 5.6] can be now formulated in the following way.

Corollary 3.10. *Skew-symmetric series of type $\left(n, \frac{1}{x^n\alpha(x)}\right)$ that also solve GCYBE exist only for $n = 0, 1$ and $n = 2$ with $\alpha_0 = 0$.*

4 Quasi-Lie bialgebra structures on $\mathfrak{g}[\![x]\!]$

We remind that F is a discrete algebraically closed field of characteristic 0 and $\mathfrak{g}[\![x]\!]$ is an F-Lie algebra equipped with the (x)-adic topology.

As we now know, series of type $(n, 1/(x^n\alpha(x)))$ solving CYBE Eq. (29) are in bijection with Lagrangian subalgebras $W\subset L(n,\alpha)$ complementary to the diagonal. On the other hand, such Lagrangian subalgebras are in bijection with non-degenerate topological Lie bialgebra structures. See [1] for their definition and classification.

It turns out, that if we drop the condition on W being a subalgebra, we get so called (non-degenerate) topological quasi-Lie bialgebras. This section is devoted to their classification up to topological twists and coordinate transformations.

Definition 4.1. A *topological quasi-Lie bialgebra* structure on $\mathfrak{g}[\![x]\!]$ consists of

- a skew-symmetric continuous linear map $\delta\colon \mathfrak{g}[\![x]\!]\to(\mathfrak{g}\otimes\mathfrak{g})[\![x,y]\!]$ and
- a skew-symmetric element $\varphi\in(\mathfrak{g}\otimes\mathfrak{g}\otimes\mathfrak{g})[\![x,y,z]\!]$,

which are subject to the following conditions

1. $\delta([a,b]) = [a\otimes 1 + 1\otimes a, \delta(b)] - [b\otimes 1 + 1\otimes b, \delta(a)]$, i.e. δ is a 1-cocycle;
2. $\frac{1}{2}\mathrm{Alt}((\delta\otimes 1)\delta(a)) = [a\otimes 1\otimes 1 + 1\otimes a\otimes 1 + 1\otimes 1\otimes a, \varphi]$;
3. $\mathrm{Alt}((\delta\otimes 1\otimes 1)\varphi) = 0$,

where $\mathrm{Alt}(x_1\otimes\ldots\otimes x_n) \coloneqq \sum_{\sigma\in S_n}\mathrm{sgn}(\sigma)x_{\sigma(1)}\otimes\ldots\otimes x_{\sigma(n)}$. ◊

Lemma 4.2. *There is a one-to-one correspondence between triples $(L, \mathfrak{g}[\![x]\!], W)$, where $(L, \mathfrak{g}[\![x]\!])$ is a topological Manin pair and $W \subset L$ is a Lagrangian subspace satisfying $W \dotplus \mathfrak{g}[\![x]\!] = L$, and quasi-Lie bialgebra structures on $\mathfrak{g}[\![x]\!]$.*

Proof. We start with a topological Manin pair $(L, \mathfrak{g}[\![x]\!])$. If $W\subset L$ is a Lagrangian subspace complementary to $\mathfrak{g}[\![x]\!]$, then it is easy to see that $W \cong \mathfrak{g}[\![x]\!]'$. Therefore, we have an isomorphism of vector spaces

$$L \cong \mathfrak{g}[\![x]\!] \dotplus \mathfrak{g}[\![x]\!]'.$$

The form on L under this isomorphism becomes standard evaluation form $\langle -,-\rangle$ on $\mathfrak{g}[\![x]\!] \dotplus \mathfrak{g}[\![x]\!]'$. We fix such an isomorphism.

Let us define two linear functions

$$p_1\colon \mathfrak{g}[\![x]\!]'\otimes\mathfrak{g}[\![y]\!]' \to \mathfrak{g}[\![x]\!] \quad\text{and}\quad p_2\colon \mathfrak{g}[\![x]\!]'\otimes\mathfrak{g}[\![y]\!]' \to \mathfrak{g}[\![x]\!]'$$

by $[f,g] = p_1(f\otimes g) + p_2(f\otimes g)$. We put

$$\delta \coloneqq p_2^\vee\colon (\mathfrak{g}[\![x]\!]')^\vee \cong \mathfrak{g}[\![x]\!] \to (\mathfrak{g}[\![x]\!]'\otimes\mathfrak{g}[\![y]\!]')^\vee \cong (\mathfrak{g}\otimes\mathfrak{g})[\![x,y]\!],$$

and let $\psi \in (\mathfrak{g}\otimes\mathfrak{g}\otimes\mathfrak{g})[\![x,y,z]\!]$ be the unique element satisfying the condition

$$\langle h, [f,g]\rangle = \langle h, p_1(f\otimes g)\rangle = \langle f\otimes g\otimes h, \psi\rangle \quad\text{for all } f,g,h\in\mathfrak{g}[\![x]\!]'. \tag{32}$$

The skew-symmetry of p_2 implies the skew-symmetry of δ, whereas the skew-symmetry of p_1 and the invariance of the evaluation form yield the skew-symmetry of ψ.

Next, we observe that for all $a,b\in\mathfrak{g}[\![x]\!]$ and $f,g\in\mathfrak{g}[\![x]\!]'$ we have

$$\begin{aligned}
&\langle [a,f], g\rangle = \langle a, [f,g]\rangle = \langle a, p_2(f\otimes g)\rangle = \langle \delta(a), f\otimes g\rangle = \langle (f\otimes 1)\delta(a), g\rangle,\\
&\langle [a,f], b\rangle = -\langle f, [a,b]\rangle = -\langle f\circ \mathrm{ad}_a, b\rangle.
\end{aligned}$$

In other words, the invariance of the form forces the following equality to hold

$$[a,f] = -f\circ\mathrm{ad}_a + (f\otimes 1)\delta(a). \tag{33}$$

Using Eq. (33) and non-degeneracy of the form we show that δ is a 1-cocycle:

$$\begin{aligned}
\langle \delta([a,b]), f\otimes g\rangle &= \langle [a,b], p_2(f\otimes g)\rangle = \langle [a,b],[f,g]\rangle = \langle [[a,b],f],g\rangle = \langle -[[b,f],a] - [[f,a],b], g\rangle\\
&= \langle [f\circ\mathrm{ad}_b - (f\otimes 1)\delta(b), a] - [f\circ\mathrm{ad}_a - (f\otimes 1)\delta(a), b], g\rangle\\
&= -\langle a, [f\circ\mathrm{ad}_b, g]\rangle + \langle b, [f\circ\mathrm{ad}_a, g]\rangle + \langle (f\otimes\mathrm{ad}_a)\delta(b), g\rangle - \langle (f\otimes\mathrm{ad}_b)\delta(a), g\rangle\\
&= \langle [a\otimes 1 + 1\otimes a, \delta(b)] - [b\otimes 1 + 1\otimes b, \delta(a)], f\otimes g\rangle.
\end{aligned} \tag{34}$$

The 1-cocycle condition implies that δ is continuous as it was noted in [1, Remark 3.16].

For conditions 2 and 3 from the definition of a topological quasi-Lie bialgebra consider the Jacobi identity for $f,g,h\in\mathfrak{g}[\![x]\!]'$:

$$\begin{aligned}
0 =&[p_1(f\otimes g),h] + [p_1(g\otimes h),f] + [p_1(h\otimes f),g]\\
&+ p_1(p_2(f\otimes g)\otimes h) + p_1(p_2(g\otimes h)\otimes f) + p_1(p_2(h\otimes f)\otimes g)\\
&+ p_2(p_2(f\otimes g)\otimes h) + p_2(p_2(g\otimes h)\otimes f) + p_2(p_2(h\otimes f)\otimes g).
\end{aligned} \tag{35}$$

We denote by $\circlearrowleft$ the summation over circular permutations of symbols f,g and h, e.g. $\circlearrowleft \langle p_1(f\otimes g),h\rangle = \langle p_1(f\otimes g),h\rangle + \langle p_1(g\otimes h),f\rangle + \langle p_1(h\otimes f),g\rangle$. Applying $\langle -,a\rangle$ to Eq. (35) for an arbitrary $a\in\mathfrak{g}[\![x]\!]$ gives

$$\begin{aligned}
\langle p_2(p_2\otimes 1)(\circlearrowleft f\otimes g\otimes h), a\rangle &= -\langle \circlearrowleft [p_1(f\otimes g),h], a\rangle\\
\langle p_2\otimes 1(\circlearrowleft f\otimes g\otimes h), \delta(a)\rangle &= \circlearrowleft \langle -h\circ\mathrm{ad}_a, p_1(f\otimes g)\rangle\\
\langle \circlearrowleft f\otimes g\otimes h, (\delta\otimes 1)\delta(a)\rangle &= \circlearrowleft \langle f\otimes g\otimes(-h\circ\mathrm{ad}_a), \psi\rangle\\
\langle f\otimes g\otimes h, \mathrm{Alt}((\delta\otimes 1)\delta(a))/2\rangle &= -\langle f\otimes g\otimes h, [1\otimes 1\otimes a + 1\otimes a\otimes 1 + a\otimes 1\otimes 1, \psi]\rangle,
\end{aligned}$$

where the very last identity holds because of the skew-symmetry of ψ. Multiplying this equality by 2 we get the relation

$$\langle f\otimes g\otimes h, \mathrm{Alt}((\delta\otimes 1)\delta(a)) + 2[1\otimes 1\otimes a + 1\otimes a\otimes 1 + a\otimes 1\otimes 1, \psi]\rangle = 0.$$

Letting $\varphi := -\psi$ we obtain the second identity from the definition of a topological quasi-Lie bialgebra structure. Applying instead $\langle s, -\rangle$, $s\in\mathfrak{g}[\![x]\!]'$ to the Jacobi identity Eq. (35) we get the desired

$$\mathrm{Alt}((\delta\otimes 1\otimes 1)\psi) = 0.$$

Therefore, $(\mathfrak{g}[\![x]\!], \delta, \varphi)$ is a topological quasi-Lie bialgebra.

For the converse direction, we put $L := \mathfrak{g}[\![x]\!] \dotplus \mathfrak{g}[\![x]\!]'$ with the standard evaluation form; we let p_1 be the unique element in $\mathrm{Hom}_{F-\mathrm{Vect}}(\mathfrak{g}[\![x]\!]'\otimes\mathfrak{g}[\![x]\!]', \mathfrak{g}[\![x]\!])$ satisfying Eq. (32) with $\psi := -\varphi$; we define $p_2 := \delta'$, i.e. the dual map of δ. The Lie bracket between two elements in $\mathfrak{g}[\![x]\!]'$ is given by the sum $p_1 + p_2$. Defining $[a, f]$ as in Eq. (33) the evaluation form becomes invariant and we get a topological Manin pair $(L, \mathfrak{g}[\![x]\!])$ with the Lagrangian subspace $\mathfrak{g}[\![x]\!]'$. These constructions are clearly inverse to each other. ■

Combining the classification of Manin pairs mentioned in Section 2 with Corollary 3.7 and Lemma 4.2 we get the following description of all topological quasi-Lie bialgebra structures on $\mathfrak{g}[\![x]\!]$.

Lemma 4.3. *There is a bijection between topological quasi-Lie bialgebra structures on $\mathfrak{g}[\![x]\!]$ and Lagrangian subspaces $W\subset L(n,\alpha)$ or $L(\infty)$ complementary to the diagonal Δ, where $\alpha = (\alpha_i \in F \mid -\infty < i \leqslant n-2)$ is an arbitrary sequence and $n\geqslant 0$. Moreover, such Lagrangian subspaces $W\subset L(n,\alpha)$ are in bijection with skew-symmetric sequences of type $(n, 1/(x^n\alpha(x)))$.*

In view of this result we call skew-symmetric series of type (n, s) as well as their projections onto the first component *quasi-r-matrices*. Quasi-Lie bialgebra structures can also be described using their associated quasi-r-matrices in the following way.

Proposition 4.4. *Assume $(\mathfrak{g}[\![x]\!], \delta, \varphi)$ is a topological quasi-Lie bialgebra and let $r\in L_2(n,\alpha)$ be the corresponding quasi-r-matrix given by the bijection from Lemma 4.3. Under the identification $\mathfrak{g}[\![(x,[x])]\!]\cong\mathfrak{g}[\![x]\!]$ we have the following identities:*

- $[a\otimes 1 + 1\otimes a, r] = -\delta(a)$ *for any* $a\in\mathfrak{g}[\![x]\!]$ *and*
- $\mathrm{CYB}(r) = -\varphi$.

The same is true for the projection $r\in(\mathfrak{g}\otimes\mathfrak{g})(\!(x)\!)[\![y]\!]$.

Proof. We start, as in the proof of Lemma 4.2, by fixing an identification $L(n,\alpha) = \Delta\dotplus W(r)\cong\mathfrak{g}[\![x]\!]\dotplus\mathfrak{g}[\![x]\!]'$. Let $\{v_{k,i}\}$ be a basis for $\mathfrak{g}[\![x]\!]'$ dual to $\{\varepsilon_{k,i} := b_i y^k\}$. Then $r = \sum_{k\geqslant 0}\sum_{i=1}^d v_{k,i}\otimes\varepsilon_{k,i}$ and we have

$$\begin{aligned}[a\otimes 1 + 1\otimes a, r] &= \sum_{k\geqslant 0}\sum_{i=1}^d [a, v_{k,i}]\otimes\varepsilon_{k,i} + v_{k,i}\otimes[a,\varepsilon_{k,i}]\\ &= \sum_{k\geqslant 0}\sum_{i=1}^d (-v_{k,i}\circ\mathrm{ad}_a + (v_{k,i}\otimes 1)\delta(a))\otimes\varepsilon_{k,i} + v_{k,i}\otimes[a,\varepsilon_{k,i}].\end{aligned}$$

Applying $\langle v_{\ell,j}\otimes v_{m,t}, -\rangle$ to the equality above we get

$$\begin{aligned}\langle v_{\ell,j}\otimes v_{m,t}, [a\otimes 1 + 1\otimes a, r]\rangle &= \sum_{k\geqslant 0}\sum_{i=1}^d \langle v_{\ell,j}\otimes v_{m,t}, (v_{k,i}\otimes 1)\delta(a)\otimes\varepsilon_{k,i}\rangle\\ &= \langle v_{\ell,j}, (v_{m,t}\otimes 1)\delta(a)\rangle\\ &= \langle v_{\ell,j}\otimes v_{m,t}, -\delta(a)\rangle.\end{aligned}$$

Applying instead $\langle\varepsilon_{\ell,j}\otimes v_{m,t}, -\rangle$ to the same equality we obtain

$$\begin{aligned}\langle\varepsilon_{\ell,j}\otimes v_{m,t}, [a\otimes 1 + 1\otimes a, r]\rangle &= \sum_{k\geqslant 0}\sum_{i=1}^d \langle\varepsilon_{\ell,j}\otimes v_{m,t}, (-v_{k,i}\circ\mathrm{ad}_a)\otimes\varepsilon_{k,i} + v_{k,i}\otimes[a,\varepsilon_{k,i}]\rangle\\ &= -\langle\varepsilon_{\ell,j}, v_{m,t}\circ\mathrm{ad}_a\rangle + \langle v_{m,t}, [a,\varepsilon_{\ell,j}]\rangle\\ &= 0.\end{aligned}$$

This implies the desired equality $[a\otimes 1 + 1\otimes a, r] = -\delta(a)$. The identity $\mathrm{CYB}(r) = -\varphi$ follows from the skew-symmetry of r, Theorem 3.6 and the fact that $\varphi = -\psi$ according to the proof of Lemma 4.2. ■

Remark 4.5. Assume $r \in (\mathfrak{g} \otimes \mathfrak{g})((x))[\![y]\!]$ is a series such that

$$[f(x) \otimes 1 + 1 \otimes f(y), r(x,y)] \in (\mathfrak{g} \otimes \mathfrak{g})[\![x,y]\!] \tag{36}$$

for all $f \in \mathfrak{g}[\![x]\!]$. Write $r = s(x^{-1},y) + g(x,y)$, where $s \in x^{-1}(\mathfrak{g}\otimes\mathfrak{g})[x^{-1}][\![y]\!]$ and $g \in (\mathfrak{g}\otimes\mathfrak{g})[\![x,y]\!]$. Then, because of Eq. (36), we must have

$$[a \otimes 1 + 1 \otimes a, s(x^{-1}, y)] = 0$$

for all $a \in \mathfrak{g}$. Since the $\mathfrak{g}$-invariant elements of $\mathfrak{g}\otimes\mathfrak{g}$ are precisely the multiples of the quadratic Casimir element Ω, we have the identity $s(x^{-1},y) = p(x^{-1},y)\Omega$ for some $p \in x^{-1}F[x^{-1}][\![y]\!]$. Furthermore, the condition

$$[ax \otimes 1 + 1 \otimes ay, p(x^{-1},y)\Omega] = [a(x-y) \otimes 1, p(x^{-1},y)\Omega] \in (\mathfrak{g} \otimes \mathfrak{g})[\![x,y]\!]$$

implies $(x-y)p(x^{-1},y) \in F[\![x,y]\!]$, meaning that there exists an $s \in F[\![y]\!]$ such that $p(x^{-1},y) = s(y)/(x-y)$. In other words, r has the form Eq. (20). This result can be considered as another motivation to study series of type (n,s). ◊

Observe that if we know one Lagrangian subspace W_0 inside $L \cong \mathfrak{g}[\![x]\!] \dotplus \mathfrak{g}[\![x]\!]'$ then any other Lagrangian subspace can be constructed from W_0 through twisting. More precisely, if $s = \sum_i a_i \otimes b^i \in (\mathfrak{g} \otimes \mathfrak{g})[\![x,y]\!]$ is a skew-symmetric tensor, then we can associate with it a (twisted) Lagrangian subspace

$$W_s := \left\{ \sum_i B(b^i, w)a_i - w \mid w \in W \right\} \subseteq L \tag{37}$$

complementary to $\mathfrak{g}[\![x]\!]$. The converse is also true; for proof see [3]. In other words, the following statement holds.

Lemma 4.6. *There is a bijection between Lagrangian subspaces $W \subseteq L(n,\alpha)$ or $L(\infty)$ and skew-symmetric tensors in $(\mathfrak{g} \otimes \mathfrak{g})[\![x,y]\!]$.*

Combining Proposition 4.4, Eq. (37) and the algorithm for constructing a quasi-r-matrix from a Lagrangian subspace $W \subset L(n,\alpha)$, $W \dotplus \Delta = L(n,\alpha)$, we obtain the following twisting rules for Lagrangian subspaces, quasi-Lie bialgebra structures and quasi-r-matrices.

Lemma 4.7. *Let $(\mathfrak{g}[\![x]\!], \delta, \varphi)$ be a topological quasi-Lie bialgebra structure corresponding to the quasi-r-matrix r. If we twist $W(r)$ with a skew-symmetric tensor s as described in Eq. (37) we obtain another topological quasi-Lie bialgebra $(\mathfrak{g}[\![x]\!], \delta_s, \varphi_s)$, such that*

1. $W(r)_s = W(r-s)$;
2. $\delta_s = \delta + ds$;
3. $\varphi_s = \varphi + \mathrm{CYB}(s) - \frac{1}{2}\mathrm{Alt}((\delta \otimes 1)s)$,

where $ds(a) := [a \otimes 1 + 1 \otimes a, s]$.

Remark 4.8. Since any quasi-r-matrix r defines a topological quasi-Lie bialgebra structure $\delta(a) = [a\otimes 1 + 1\otimes a, r]$ on $\mathfrak{g}[\![x]\!]$, the third condition in Definition 4.1 is trivially satisfied. In other words,

$$\mathrm{Alt}((\delta \otimes 1 \otimes 1)\mathrm{CYB}(r)) = 0$$

for any quasi-r-matrix r. ◊

Lemma 4.6 and Lemma 4.7 state that, in order to obtain a description of topological quasi-Lie bialgebra structures on $\mathfrak{g}[\![x]\!]$ up to twisting it is enough to find a single Lagrangian subspace W_0, complementary to $\mathfrak{g}[\![x]\!]$, inside $L(\infty)$ and each $L(n,\alpha)$. The same is true for the associated quasi-r-matrices

The case $L(\infty)$ is trivial, because by definition $\mathfrak{g}[\![x]\!]' = \bigoplus_{j\geqslant 0} \mathfrak{g} \otimes a_j \subseteq L(\infty)$ is a Lagrangian subalgebra (see Definition 2.1). Similar to the Lie bialgebra case, topological quasi-Lie bialgebras corresponding to the Manin pair $(L(\infty), \mathfrak{g}[\![x]\!])$ are called *degenerate*.

Let us now focus on *non-degenerate* topological quasi-Lie bialgebra structures, i.e. the ones corresponding to the Manin pairs $(L(n,\alpha),\Delta)$. By Proposition 2.5 for each Manin pair $(L(n,\alpha),\Delta)$ there exists an appropriate coordinate transformation that makes it into $(L(n,\beta),\Delta)$, where $\beta_0 = \alpha_0$ and all other $\beta_i = 0$. This means, that to classify all non-degenerate topological quasi-Lie bialgebras on $\mathfrak{g}[\![x]\!]$, up to coordinate transformations and twisting, it is enough to construct a Lagrangian subspace W_0 within each $L(n,\alpha_0) := L(n,(\dots,0,\alpha_0,0,\dots,0))$ complementary to Δ. Equivalently, it is enough to find a quasi-r-matrix of type (n,α_0) for any $n \geqslant 0$ and $\alpha_0 \in F$.

4.1 Lagrangian subspaces of $L(n, \alpha_0)$

As before we let $\{b_i\}_{i=1}^d$ be an orthonormal basis for $\mathfrak{g}$ with respect to the Killing form κ. The form B on $L(n, \alpha_0)$ has the following explicit form

$$B(a \otimes (f, [p]), b \otimes (g, [q])) = \begin{cases} \kappa(a,b)\,\{\mathrm{coeff}_{n-1}(fg - pq) - \alpha_0 \mathrm{coeff}_0(fg - pq)\} & \text{if } n \geqslant 2, \\ \kappa(a,b)\mathrm{coeff}_{n-1}(fg - pq) & \text{if } n = 0, 1. \end{cases} \tag{38}$$

We now present an explicit construction for a Lagrangian subspace of $L(n, \alpha_0)$ complementary to Δ for arbitrary $n \geqslant 0$ and $\alpha_0 \in F$. Using the twisting procedure from Lemma 4.7, this subspace can be twisted in order to obtain all other Lagrangian subspaces of $L(n, \alpha_0)$ complementary to Δ.

n = 0: When $n = 0$, the subalgebra $W_0 := x^{-1}\mathfrak{g}[\![x^{-1}]\!] \subseteq \mathfrak{g}(\!(x)\!)$ is known to be Lagrangian.

n = 1: For $n = 1$ it is easy to see that the subspace

$$W_0 := \mathrm{span}_F\{b_i(1, -1), b_i(x^{-k}, 0) \mid k \geqslant 1,\ 1 \leqslant i \leqslant d\} \subset L(1, \alpha_0) \tag{39}$$

is Lagrangian and complementary to the diagonal Δ.

n = 2k: For even $n \geqslant 2$ and arbitrary $\alpha_0 \in F$ the subspace $W_0 \subset L(n, \alpha_0)$ spanned by the elements

$$b_i\left\{(x^{(n-1)-m}, 0) - \alpha_0(x^{2(n-1)-m}, 0) + \alpha_0^2(x^{3(n-1)-m}, 0) - \alpha_0^3(x^{4(n-1)-m}, 0) + \dots\right\},\ 0 \leqslant m \leqslant \frac{n}{2} - 1,$$
$$b_i(0, -[x]^{(n-1)-\ell}),\ \frac{n}{2} \leqslant \ell < n - 1,$$
$$b_i(0, -1 + \frac{\alpha_0}{2}[x]^{n-1}),$$
$$b_i(x^{-k}, 0), k \geqslant 1,$$

is Lagrangian and complementary to the diagonal.

n = 2k + 1: Modifying slightly the basis for even case we obtain the following basis for $W_0 \subset L(n, \alpha_0)$ with odd $n \geqslant 3$:

$$b_i\left\{(x^{(n-1)-m}, 0) - \alpha_0(x^{2(n-1)-m}, 0) + \alpha_0^2(x^{3(n-1)-m}, 0) - \alpha_0^3(x^{4(n-1)-m}, 0) + \dots\right\},\ 0 \leqslant m \leqslant \frac{n-1}{2} - 1,$$
$$b_i\left\{(x^{\frac{n-1}{2}}, -[x]^{\frac{n-1}{2}}) - \alpha_0(x^{\frac{3(n-1)}{2}}, 0) + \alpha_0^2(x^{\frac{5(n-1)}{2}}, 0) - \alpha_0^3(x^{\frac{7(n-1)}{2}}, 0) + \dots\right\},$$
$$b_i(0, -[x]^{(n-1)-\ell}),\ \frac{n-1}{2} + 1 \leqslant \ell < n - 1,$$
$$b_i(0, -1 + \frac{\alpha_0}{2}[x]^{n-1}),$$
$$b_i(x^{-k}, 0), k \geqslant 1.$$

The subspaces above were constructed by "guessing". However, there is an abstract procedure that produces Lagrangian subspaces for arbitrary n and α. We present it here for completeness.

The easiest skew-symmetric (n, s)-type series is given by

$$\begin{aligned} r(x, y) &:= \frac{1}{2}\left(\frac{s(x)y^n\Omega}{x - y} + \frac{s(y)x^n\Omega}{x - y}\right) = \frac{s(x)y^n\Omega}{x - y} + \frac{\Omega}{2}\left(\frac{s(y)x^n - s(x)y^n}{x - y}\right) \\ &= \frac{s(x)y^n\Omega}{x - y} - \frac{1}{2}\sum_{k,\ell=0}^{\infty}\sum_{i,j=1}^{d} B(sw_{k,i}, sw_{\ell,j})b_i(x, [x])^k \otimes b_j(y, [y])^\ell, \end{aligned}$$

where we recall that

$$B(sw_{k,i}, sw_{\ell,j}) = \begin{cases} -s_{k+\ell-n+1} & \text{if } i = j,\ 0 \leqslant k, \ell \leqslant n - 1 \text{ and } k + \ell \geqslant n - 1, \\ s_{k+\ell-n+1} & \text{if } i = j \text{ and } k\,, \ell \geqslant n, \\ 0 & \text{otherwise.} \end{cases}$$

By Corollary 3.7 the subspace

$$W(r) = \operatorname{span}_F \left\{ sw_{k,i} - \frac{1}{2}\sum_{\ell=0}^{\infty} B(sw_{\ell,i}, sw_{k,i}) b_i(x,[x])^\ell \,\middle|\, k \geqslant 0, 1 \leqslant d \leqslant n \right\}$$
$$= \operatorname{span}_F \left\{ sw_{k,i} + \frac{1}{2}\left(\sum_{\ell=0}^{n-1} s_{k+\ell-n+1} b_i(x,[x])^\ell - \sum_{\ell=n}^{\infty} s_{k+\ell-n+1} b_i(x,[x])^\ell \right) \,\middle|\, k \geqslant 0, 1 \leqslant d \leqslant n \right\}$$

is Lagrangian and complementary to the diagonal. Here we used the convention that $s_k = 0$ for $k < 0$. Calculating the basis explicitly for some particular s requires some effort and it may not look as friendly as the ones given above.

4.2 Quasi-r-matrices

The goal of this section is to describe the quasi-r-matrices corresponding to the Lagrangian subspaces described in the previous section. The twisting procedure from Lemma 4.7 then yields all other quasi-r-matrices.

The proof of Theorem 3.6 gives us an algorithm for constructing a series of type $(n, s(x) := 1/(x^n\alpha(x)))$ from a subspace $W \subset L(n,\alpha)$ complementary to the diagonal. More precisely, the desired series is given by

$$\sum_{k\geqslant 0}\sum_{i=1}^{d} v_{k,i} \otimes b_i(y^k,[y]^k), \tag{40}$$

where

$$W = \operatorname{span}_F\{v_{k,i} \mid k \geqslant 0,\ 1 \leqslant i \leqslant d\} \text{ and } B(v_{k,i}, b_j(y^\ell,[y]^\ell)) = \delta_{i,j}\delta_{k,\ell},$$

i.e. $\{v_{k,i}\}$ is a basis of V dual to $\{b_i(y^k,[y]^k)\}$. Indeed, non-degeneracy of the form B then implies that $v_{k,i}$ has the desired form $v_{k,i} = sw_{k,i} + g_{k,i}$ for some $g_{k,i} \in (\mathfrak{g}\otimes\mathfrak{g})[\![x,y]\!]$.

Applying this idea to W_0's constructed in the preceding section we get the following series.

n = 0: The classical r-matrix (equivalently $(0,1)$-type series) corresponding to $W_0 := x^{-1}\mathfrak{g}[\![x^{-1}]\!] \subseteq \mathfrak{g}(\!(x)\!)$ is the Yang's matrix $\Omega/(x-y)$.

n = 1: The quasi-r-matrix corresponding to $\operatorname{span}_F\{b_i(1,-1), b_i(x^{-k},0) \mid k \geqslant 1,\ 1 \leqslant i \leqslant d\} \subset L(1,\alpha_0)$ is

$$\frac{y\Omega}{x-y} + \frac{1}{2}\sum_{i=1}^{d} b_i(1,-1)\otimes b_i(1,1) \in L_2(1,1) \text{ with the projection } \frac{y\Omega}{x-y} + \frac{1}{2}\Omega \in (\mathfrak{g}\otimes\mathfrak{g})(\!(x)\!)[\![y]\!].$$

n = 2k: For even $n \geqslant 2$ and arbitrary $\alpha_0 \in F$ we have the following quasi-r-matrix

$$\frac{1}{1+\alpha_0 x^{n-1}}\frac{y^n\Omega}{x-y} + \frac{\Omega}{1+\alpha_0 x^{n-1}} \sum_{0\leqslant m<\frac{n}{2}} x^{(n-1)-m}y^m$$
$$+ \frac{\alpha_0\Omega}{(1+\alpha_0x^{n-1})(1+\alpha_0y^{n-1})}\left(y^{2(n-1)} + \sum_{\frac{n}{2}\leqslant\ell<n-1} x^{(n-1)-\ell}y^{(n-1)+\ell} - \frac{1}{2}x^{n-1}y^{n-1} \right).$$

n = 2k + 1: In the odd case $n \geqslant 3$ the series corresponding to $W_0 \subset L(n,\alpha_0)$ is

$$\frac{1}{1+\alpha_0 x^{n-1}}\frac{y^n\Omega}{x-y} + \frac{\Omega}{1+\alpha_0 x^{n-1}} \left(x^{\frac{n-1}{2}}y^{\frac{n-1}{2}} + \sum_{0\leqslant m<\frac{n-1}{2}} x^{(n-1)-m}y^m \right)$$
$$+ \frac{\alpha_0\Omega}{(1+\alpha_0x^{n-1})(1+\alpha_0y^{n-1})}\left(y^{2(n-1)} + \sum_{\frac{n-1}{2}<\ell<n-1} x^{(n-1)-\ell}y^{(n-1)+\ell} - \frac{1}{2}x^{n-1}y^{n-1} \right).$$

5 Lie algebra splittings of $L(n,\alpha)$ and generalized r-matrices

By Corollary 3.7 we have a bijection between subalgebras of $L(n,\alpha)$ and series of type $(n, 1/(x^n\alpha(x)))$ solving GCYBE. Therefore, we can construct new solutions to GCYBE by finding subalgebras of $L(n,\alpha)$ complementary to the diagonal. However, as the following result shows, the most interesting new solutions should arise from unbounded subalgebras of $L(n,\alpha)$, $n > 2$.

Proposition 5.1. *Let $L(n,\alpha) = \Delta \dotplus W$ for some subalgebra $W \subset L(n,\alpha)$ and $n > 2$. Assume W is bounded, i.e. there is an integer $N > 0$ such that*

$$x^{-N}\mathfrak{g}[x^{-1}] \subseteq W_+ \subseteq x^N\mathfrak{g}[x^{-1}],$$

where W_+ is the projection of $W \subset L(n,\alpha) = \mathfrak{g}((x)) \oplus \mathfrak{g}[x]/x^n\mathfrak{g}[x]$ on the first component $\mathfrak{g}((x))$. Then there is an element $\sigma \in \mathrm{Aut}_{F[x]-\mathrm{LieAlg}}(\mathfrak{g}[x])$ such that

$$\{0\} \times [x^2]\mathfrak{g}[x]/x^n\mathfrak{g}[x] \subseteq (\sigma \times \sigma)W \subseteq x\mathfrak{g}[x^{-1}] \times \mathfrak{g}[x]/x^n\mathfrak{g}[x]$$

and the image $\widetilde{W}$ under the canonical projection $L(n,\alpha) \to L(2,\alpha)$ is a subalgebra satisfying $L(2,\alpha) = \Delta \dotplus \widetilde{W}$.

In the language of (n,s)-type series: Let

$$r = \frac{s(x)y^n\Omega}{x-y} + g(x,y)$$

be the generalized r-matrix corresponding to a bounded $W \subset L(n,\alpha), n \geqslant 2$. Then there is $p(x,y) \in (\mathfrak{g}\otimes\mathfrak{g})[x,y]$ of degree at most one in x and an element $\sigma \in \mathrm{Aut}_{F[x]-\mathrm{LieAlg}}(\mathfrak{g}[x])$ such that

$$(\sigma(x)\otimes\sigma(y))r(x,y) = y^{n-2}\Big(\underbrace{\frac{s(x)y^2\Omega}{x-y} + p(x,y)}_{r'(x,y)}\Big),$$

where r' is a generalized r-matrix in $L_2(2,\alpha)$.

Proof. The condition $x^{-N}\mathfrak{g}[x^{-1}] \subseteq W_+ \subseteq x^N\mathfrak{g}[x^{-1}]$ means exactly that W_+ is an order. Moreover, since W is complementary to the diagonal, we have $W_+ + \mathfrak{g}[x] = \mathfrak{g}[x,x^{-1}]$. It was shown in [11] that such orders, up to the action of some $\sigma \in \mathrm{Aut}_{F[x]-\mathrm{LieAlg}}(\mathfrak{g}[x])$, are contained in a maximal order $\mathfrak{M}$ associated to the so called fundamental simplex Δ_{st}. These maximal orders are explicitly described in [11] and satisfy $\mathfrak{M} \subseteq x\mathfrak{g}[x^{-1}]$. Therefore, we have $\sigma W_+ \subseteq \mathfrak{M} \subseteq x\mathfrak{g}[x^{-1}]$. Moreover, we have the identity

$$(\sigma\times\sigma)W \dotplus \Delta = L(n,\alpha),$$

implying the inclusion $\{0\} \times [x^2]\mathfrak{g}[x]/x^n\mathfrak{g}[x] \subseteq (\sigma\times\sigma)W$. The remaining parts follow straightforward from the construction Theorem 3.6. ■

Unfortunately, we have not found a new example of an unbounded subalgebra of $L(n,\alpha)$. However, we present an infinite family of bounded subalgebras. We believe these examples are still interesting because their orthogonal complements, which are important in the view of Adler-Kostant-Symes scheme, are unbounded if $\alpha \neq 0$.

Consider the subspaces of $L(n,\alpha_0)$, $n > 0$:

$$\begin{aligned} W_0 &= \mathrm{span}_F\{b_i(x^{-k},0), b_i(1,0), b_i(0,-[x]^\ell) \mid k \geqslant 1,\ 1 \leqslant \ell \leqslant n-1\},\\ W_1 &= \mathrm{span}_F\{b_i(x^{-k},0), b_i(0,-1), b_i(0,-[x]^\ell) \mid k \geqslant 1,\ 1 \leqslant \ell \leqslant n-1\}. \end{aligned}$$

These are clearly subalgebras. The corresponding generalizerd r-matrices are

$$\begin{aligned} r_0 &= \frac{1}{1+\alpha_0x^{n-1}}\frac{y^n\Omega}{x-y} + \frac{y^{n-1}\Omega}{(1+\alpha_0x^{n-1})(1+\alpha_0y^{n-1})}\\ &\quad+ \frac{\alpha_0\Omega}{(1+\alpha_0x^{n-1})(1+\alpha_0y^{n-1})}\left(y^{2(n-1)} + \sum_{0\leqslant\ell<n-1} x^{(n-1)-\ell}y^{(n-1)+\ell}\right)\\ &= \frac{y^{n-1}}{1+\alpha_0y^{n-1}}\left(\frac{y\Omega}{x-y} + \Omega\right),\\ r_1 &= \frac{1}{1+\alpha_0x^{n-1}}\frac{y^n\Omega}{x-y} + \frac{\alpha_0\Omega}{(1+\alpha_0x^{n-1})(1+\alpha_0y^{n-1})}\left(y^{2(n-1)} + \sum_{0<\ell<n-1} x^{(n-1)-\ell}y^{(n-1)+\ell}\right)\\ &= \frac{1}{1+\alpha_0y^{n-1}}\frac{y^n\Omega}{x-y}. \end{aligned}$$

By considering decompositions $\mathfrak{g} = \mathfrak{s}_1 \dotplus \mathfrak{s}_2$ of $\mathfrak{g}$ into direct sums of subalgebras we can get an infinite family of generalized r-matrices "in between" r_0 and r_1. More precisely, let $\{s_{1,i}\}_{i=1}^{d_1}$ and $\{s_{2,j}\}_{j=1}^{d_2}$ be bases for $\mathfrak{s}_1$ and $\mathfrak{s}_2$ respectively. Such a decomposition leads to another subalgera of $L(n,\alpha_0)$:

$$\begin{aligned} W_{01} \coloneqq \mathrm{span}_F\Big\{&b_i(x^{-k},0), s_{1,m}(1,0), s_{2,j}(0,1), b_i(0,-[x]^\ell) \mid k \geqslant 1, 1 \leqslant \ell \leqslant n-1,\ 1 \leqslant i \leqslant d,\\ &1 \leqslant m \leqslant d_1,\ 1 \leqslant j \leqslant d_2\Big\}. \end{aligned}$$

Rewrite the elements b_i in terms of $s_{1,m}$ and $s_{2,j}$:

$$b_i = \sum_{m=1}^{d_1} \lambda^i_{1,m} s_{1,m} + \sum_{j=1}^{d_2} \lambda^i_{2,j} s_{2,j},$$

where $\lambda^i_{1,m}, \lambda^i_{2,j} \in F$. Finding a basis in W_{12} dual to $\{b_i(y^m, [y]^m)\} \subset \Delta$ and then projecting the generating series for W_{01} onto the first component we obtain the following generalized r-matrix

$$\begin{aligned} r_{01} &= \frac{1}{1+\alpha_0 x^{n-1}} \frac{y^n \Omega}{x-y} + \frac{\alpha_0 \Omega}{(1+\alpha_0 x^{n-1})(1+\alpha_0 y^{n-1})} \left(y^{2(n-1)} + \sum_{0<\ell<n-1} x^{(n-1)-\ell} y^{(n-1)+\ell} \right) \\ &+ \frac{y^{n-1}}{1+\alpha_0 y^{n-1}} \sum_{i=1}^{d} \sum_{m=1}^{d_1} \lambda^i_{1,m} s_{1,m} \otimes b_i. \qquad (41) \\ &= \frac{y^{n-1}}{1+\alpha_0 y^{n-1}} \left(\frac{y\Omega}{x-y} + \sum_{i=1}^{d} \sum_{m=1}^{d_1} \lambda^i_{1,m} s_{1,m} \otimes b_i \right) \end{aligned}$$

Clearly r_{01} coincides with r_0 when $\mathfrak{s}_1 = \mathfrak{g}$ and r_1 if $\mathfrak{s}_2 = \mathfrak{g}$. The corresponding orhogonal complements are

$$\begin{aligned} W_0^\perp &= W(\overline{r_0}) = \operatorname{span}_F \left\{ b_i\left(0, [x]^{n-1}\right), b_i\left(\frac{x^{-k(n-1)-m}}{1+\alpha_0 x^{n-1}}, 0 \right) \mid k \geqslant -1, 0 < m < n-1 \right\}, \\ W_1^\perp &= W(\overline{r_1}) = \operatorname{span}_F \left\{ b_i\left(\frac{x^{-k(n-1)-m}}{1+\alpha_0 x^{n-1}}, 0 \right) \mid k \geqslant -1, 0 \leqslant m < n-1 \right\}, \qquad (42) \\ W_{01}^\perp &= W(\overline{r_{01}}) = \mathfrak{s}_1^\perp \left(\frac{x^{n-1}}{1+\alpha_0 x^{n-1}}, 0 \right) \dotplus \mathfrak{s}_2^\perp (0, [x]^{n-1}) \\ &\quad \dotplus \operatorname{span}_F \left\{ b_i\left(\frac{x^{-k(n-1)-m}}{1+\alpha_0 x^{n-1}}, 0 \right) \mid k \geqslant -1, 0 < m < n-1 \right\}, \end{aligned}$$

which are unbounded because of the factor $1/(1+\alpha_0 x^{n-1})$.

Note that a series of type (n,s) defines a subspace inside $L(n,\alpha)$ for any α, because the subalgebra property is not affected by the form. With the previous examples in mind we can prove the following statement.

Lemma 5.2. *Let B_0 and B_α be the bilinear forms on $L(n,0)$ and $L(n,\alpha)$ respectively. For a series r of type (n,s) we have*

$$W(r)^{\perp_{B_\alpha}} = \frac{1}{x^n \alpha(x)} W(r)^{\perp_{B_0}} \subset L(n,\alpha). \qquad (43)$$

Proof. Set $u(x) \coloneqq 1/(x^n \alpha(x))$. Write

$$r = \sum_{k\geqslant 0} \sum_{i=1}^{d} (s w_{k,i} + g_{k,i}) \otimes b_i(y^k, [y]^k) \quad \text{and} \quad \overline{r} = \sum_{k\geqslant 0} \sum_{i=1}^{d} (s w_{k,i} + \overline{g_{k,i}}) \otimes b_i(y^k, [y]^k).$$

Then by Theorem 3.6 and definition Eq. (11) $B_\alpha(sw_{k,i}+g_{k,i}, u(sw_{\ell,j}+\overline{g_{\ell,j}})) = B_0(sw_{k,i}+g_{k,i}, sw_{\ell,j}+\overline{g_{\ell,j}}) = 0$ ■

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Data availability statements

Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

Declarations

The authors have no competing interests to declare that are relevant to the content of this article.

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