SUBGROUP GROWTH IN FREE CLASS-2-NILPOTENT GROUPS

VIOLA SICONOLFI, MARLIES VANTOMME, CHRISTOPHER VOLL

Abstract. We describe an effective procedure to compute the local subgroup zeta functions of the free class-2-nilpotent groups on d generators, for all d. For $d = 4$, this yields a new, explicit formula. For $d \in \{4, 5\}$, we compute the topological subgroup zeta functions. We also obtain general results about the reduced and topological subalgebra zeta functions. For the former, we determine the behaviour at one; for the latter, the degree and behaviours at zero and infinity. Some of these results confirm, in the relevant special cases, general conjectures by Rossmann.

Contents

1. Introduction

1.1. **Setup.** We study the subgroup growth of finitely generated free nilpotent groups of nilpotency class two. For $d \in \mathbb{N}_{\geqslant 2}$, the free nilpotent group $F_{2,d}$ of nilpotency class two on d generators has presentation

$$F_{2,d} = \langle g_1, \ldots, g_d \mid \forall 1 \leqslant i, j, k \leqslant d : [[g_i, g_j], g_k] = 1 \rangle.$$

The *subgroup zeta function of* $F_{2,d}$ is the Dirichlet generating series

$$\zeta_{F_{2,d}}(s) = \sum_{H \leqslant F_{2,d}} |F_{2,d} : H|^{-s},$$

where s is a complex variable (and $\infty^{-s} = 0$, so the sum extends in effect only over the subgroups of finite index). It is well-known that $\zeta_{F_{2,d}}(s)$ has an Euler product

$$\zeta_{F_{2,d}}(s) = \prod_{p \text{ prime}} \zeta_{F_{2,d},p}(s), \tag{1.1}$$

where, for each prime p, the Euler factor $\zeta_{F_{2,d},p}(s)$ enumerates the subgroups of $F_{2,d}$ of p-power index; cf. [8, Prop. 4]. By a deep result of Grunewald, Segal, and Smith ([8,

Date: 23rd June 2023.

2020 *Mathematics Subject Classification.* 20E07, 11S40, 11M41.

Key words and phrases. subgroup growth, subgroup zeta functions, free nilpotent groups, subalgebra zeta functions, free nilpotent Lie rings.

Thm. 1]) these *local* zeta functions are all rational in the parameter $t = p^{-s}$. Computing these—and other groups'—local zeta functions explicitly, however, is difficult. Previously, explicit formulas were only known for $d \in \{2, 3\}$; see Section 7.

The free step-2-nilpotent Lie ring $\mathfrak{f}_{2,d}$ on d generators has presentation

$$\mathfrak{f}_{2,d} = \langle x_1, \ldots, x_d \mid \forall 1 \leqslant i, j, k \leqslant d : [[x_i, x_j], x_k] = 0 \rangle.$$

The *subalgebra zeta function of* $\mathfrak{f}_{2,d}$ is

$$\zeta_{\mathfrak{f}_{2,d}}(s) = \sum_{H \leqslant \mathfrak{f}_{2,d}} |\mathfrak{f}_{2,d} : H|^{-s}.$$

It is well known that the problem of counting subgroups of the group $F_{2,d}$ is the same as the problem of counting subalgebras of $\mathfrak{f}_{2,d}$. Indeed, the fact that

$$\zeta_{F_{2,d}}(s) = \zeta_{\mathfrak{f}_{2,d}}(s)$$

is not hard to verify; see [8, Chap. 4]. It justifies that we concentrate on subalgebra zeta functions in the following.

For any (commutative) ring R, we set $\mathfrak{f}_{2,d}(R) = \mathfrak{f}_{2,d} \otimes_{\mathbb{Z}} R$ and consider the subalgebra zeta function

$$\zeta_{\mathfrak{f}_{2,d}(R)}(s) = \sum_{H \leqslant \mathfrak{f}_{2,d}(R)} |\mathfrak{f}_{2,d}(R) : H|^{-s}, \tag{1.2}$$

enumerating R-subalgebras of $\mathfrak{f}_{2,d}(R)$ of finite index. In practice, we will focus on rings R which are compact discrete valuation rings (cDVRs), viz. finite extensions of the p-adic integers $\mathbb{Z}_p$ (in characteristic zero) or rings $\mathbb{F}_q[\![T]\!]$ of formal power series over finite fields (in positive characteristic). We write $\mathfrak{p}$ for the unique maximal ideal of $\mathfrak{o}$, with residue field cardinality $|\mathfrak{o}/\mathfrak{p}| =: q_{\mathfrak{o}}$, a prime power. By (1.1), the Euler product decomposition (1.1) is mirrored by the factorization

$$\zeta_{\mathfrak{f}_{2,d}}(s) = \prod_{p \text{ prime}} \zeta_{\mathfrak{f}_{2,d}(\mathbb{Z}_p)}(s). \tag{1.3}$$

1.2. **Main results.** In the present paper, we present an effective procedure to compute the local subalgebra zeta functions $\zeta_{\mathfrak{f}_{2,d}(\mathfrak{o})}(s)$ for all $d \in \mathbb{N}_{\geqslant 2}$ and cDVR's $\mathfrak{o}$. Theorem 4.24 establishes an explicit formula for $\zeta_{\mathfrak{f}_{2,d}(\mathfrak{o})}(s)$, by defining a bivariate rational function $\zeta_{\mathfrak{f}_{2,d}}(q,t)$ such that $\zeta_{\mathfrak{f}_{2,d}(\mathfrak{o})}(s) = \zeta_{\mathfrak{f}_{2,d}}(q_{\mathfrak{o}}, q_{\mathfrak{o}}^{-s})$. This rational expression $\zeta_{\mathfrak{f}_{2,d}}(q,t)$ is given in terms of Gaussian q-multinomials and finitely many generating functions enumerating the elements of a subset of $\mathbb{N}_0^m$.

We apply this formula in different ways, both theoretically and practically. First, it inspires a notion of *no-overlap subalgebra zeta function* of $\mathfrak{f}_{2,d}(\mathfrak{o})$ enumerating, loosely speaking, "most of" the finite-index subalgebras of $\mathfrak{f}_{2,d}(\mathfrak{o})$, see Section 5.1. We prove that this summand satisfies the same local functional equation as $\zeta_{\mathfrak{f}_{2,d}(\mathfrak{o})}(s)$; see Theorem 5.9. Second, we derive from the formula that, for all d and almost all cDVRs $\mathfrak{o}$, the $\mathfrak{p}$-adic subalgebra zeta function $\zeta_{\mathfrak{f}_{2,d}(\mathfrak{o})}(s)$ has a simple pole at $s = 0$, establishing a conjecture of Rossmann for the relevant algebras, see Theorem 5.13. Third, we compute the subalgebra zeta functions $\zeta_{\mathfrak{f}_{2,4}(\mathfrak{o})}(s)$ fully explicitly by implementing the formula in `SageMath` [12] using `LattE` [1] and `Zeta` [11], see Theorem 7.3 for a paraphrase and 10.5281/zenodo.7966735 for full details.

To consider Euler products such as (1.3) is just one way to capture information about "many" $\mathfrak{p}$-adic zeta functions uniformly. Others include the reduced zeta functions pioneered by Evseev ([7]) and the topological zeta functions developed by Rossmann ([10]). Both may be paraphrased as results of "setting $q_{\mathfrak{o}} = 1$", in subtly different ways. Crudely speaking, the reduced zeta function $\zeta^{\mathrm{red}}_{\mathfrak{f}_{2,d}}(t)$ is the univariate rational expression in t defined as $\zeta_{\mathfrak{f}_{2,d}}(1,t)$. Equally informally, the topological zeta function $\zeta^{\mathrm{top}}_{\mathfrak{f}_{2,d}}(s)$ is the univariate rational expression in s obtained as the first non-zero coefficient of $\zeta_{\mathfrak{f}_{2,d}}(q, q^{-s})$, expanded in $q - 1$.

In Theorem 6.8 we show that the reduced subalgebra zeta function $\zeta^{\mathrm{red}}_{\mathfrak{f}_{2,d}}(t)$ has a pole at $t = 1$ of order $D := d + \binom{d}{2}$, which is the $\mathbb{Z}$-rank of $\mathfrak{f}_{2,d}$. In Theorem 6.10 we establish that D is also the degree of the topological subalgebra zeta function $\zeta^{\mathrm{top}}_{\mathfrak{f}_{2,d}}(s)$. In Theorem 6.13 and Theorem 6.14 we show that the topological zeta function has a simple pole at $s = 0$ and compute its residue there. This confirms, in the relevant special cases, general conjectures by Rossmann. The topological and reduced subalgebra zeta functions feature together in Theorem 6.11; it links the topological zeta function's behaviour at infinity and the reduced zeta function's residue at $t = 1$. We also compute the topological subalgebra zeta functions $\zeta^{\mathrm{top}}_{\mathfrak{f}_{2,4}}(s)$ and $\zeta^{\mathrm{top}}_{\mathfrak{f}_{2,5}}(s)$ fully explicitly using our implementation of Theorem 4.24, see Theorems 7.5 and 7.7.

1.3. **Related work.** For $d \leqslant 3$, the $\mathfrak{p}$-adic subalgebra zeta functions $\zeta_{\mathfrak{f}_{2,d}(\mathfrak{o})}(s)$—and, as corollaries, their topological and reduced analogues—have been known for some time; see Section 7 for explicit formulas and references. For $c > 2$, the subalgebra zeta functions of the free step-c-nilpotent Lie rings on d generators $\mathfrak{f}_{c,d}$ are largely unknown. To our knowledge, explicit formulas are only known for $(c, d) = (3, 2)$, by work of Woodward; cf. [6, Thm. 2.35].

The ideal zeta functions $\zeta^{\triangleleft}_{\mathfrak{f}_{2,d}(\mathfrak{o})}(s)$, enumerating ideals of finite index, have been computed, for all d, in [16]. This yields, in particular, the (global) ideal zeta function $\zeta^{\triangleleft}_{\mathfrak{f}_{2,d}}(s) = \prod_p \zeta^{\triangleleft}_{\mathfrak{f}_{2,d}(\mathbb{Z}_p)}(s)$. In analogy with (1.1) we have $\zeta^{\triangleleft}_{\mathfrak{f}_{2,d}}(s) = \zeta^{\triangleleft}_{F_{2,d}}(s)$, the normal zeta function of the free class-2-nilpotent group $F_{2,d}$, enumerating normal subgroups of finite index.

1.4. **Organization and notation.** In Section 2, we recall some well-known nomenclature and results. We consider Gaussian binomial and multinomial coefficients in Section 2.1; the enumeration of submodules of $\mathfrak{o}$-modules of finite rank in Section 2.2, convex polyhedral cones in $\mathbb{Q}^m$ in Section 2.3; monoids in $\mathbb{N}_0^m$, in particular, solution sets of systems of linear homogeneous Diophantine equations, in Section 2.4; and generating functions of subsets of $\mathbb{N}_0^m$, in particular monoids, in Section 2.5. In Sections 2.6 and 2.7, we define some notation and prove some preliminary results for certain subsets of a monoid in $\mathbb{N}_0^m$. In Section 3, we define the specific monoids and subsets in $\mathbb{N}_0^m$ that are used in the later sections to write down formulas for the considered subalgebra zeta functions.

Section 4 culminates in Theorem 4.24, an explicit formula for $\zeta_{\mathfrak{f}_{2,d}(\mathfrak{o})}(s)$ as a finite sum, whose summands are products of Gaussian q-multinomials and generating functions of subsets of $\mathbb{N}_0^m$ as discussed in Section 2.5. In Section 5, we use this explicit formula to obtain several general results on the $\mathfrak{p}$-adic subalgebra zeta functions $\zeta_{\mathfrak{f}_{2,d}(\mathfrak{o})}(s)$. Notably, we introduce the no-overlap subalgebra zeta function of $\mathfrak{f}_{2,d}(\mathfrak{o})$ in Section 5.1 and show that for all d and almost all cDVRs $\mathfrak{o}$, the $\mathfrak{p}$-adic subalgebra zeta function $\zeta_{\mathfrak{f}_{2,d}(\mathfrak{o})}(s)$ has a simple pole at $s = 0$ in Section 5.4.

In Section 6, we obtain results on the reduced and topological zeta functions mentioned in Section 1.2. For the former, we determine the behaviour at $t = 1$ and for the latter, the degree and behaviours at zero and infinity. In Section 7, we record fully explicit formulas for $\mathfrak{p}$-adic, reduced and topological subalgebra zeta functions associated with $\mathfrak{f}_{2,d}$ for small values of d, both known and new.

Table 1.1 gives a partial list of the notation used.

Acknowledgements. This work forms part of the second author's doctoral dissertation, supervised by the third author. We would like to thank Tobias Rossmann for pointing

Notation	Meaning	Location
$\mathfrak{o}$	compact discrete valuation ring	Section 1.1
$q_\mathfrak{o}$	cardinality of the residue field $\mathfrak{o}/\mathfrak{p}$, a prime power	Section 1.1
$\mathfrak{f}_{2,d}(\mathfrak{o})$	tensor product $\mathfrak{f}_{2,d} \otimes_\mathbb{Z} \mathfrak{o}$	Section 1.1
$\zeta_{\mathfrak{f}_{2,d}(\mathfrak{o})}(s)$	subalgebra zeta function of $\mathfrak{f}_{2,d}(\mathfrak{o})$	(1.2)
$\zeta_{\mathfrak{f}_{2,d}}(q,t)$	rational function with $\zeta_{\mathfrak{f}_{2,d}}(q_\mathfrak{o}, q_\mathfrak{o}^{-s}) = \zeta_{\mathfrak{f}_{2,d}(\mathfrak{o})}(s)$ for all $\mathfrak{o}$	Section 1.2
D	$d + \binom{d}{2} = \binom{d+1}{2}$, the $\mathbb{Z}$-rank of $\mathfrak{f}_{2,d}$	Section 1.2
d'	$\binom{d}{2}$	Section 2.1
$\binom{n}{J}_q$	Gaussian multinomial coefficient	Definition 2.2
$\mathcal{P}_n$	set of integer partitions of at most n parts	Section 2.2.1
$\nu \leqslant \mu$	$\nu_i \leqslant \mu_i$ for $i \in [n]$ for partitions $\mu, \nu \in \mathcal{P}_n$	Section 2.2.1
$\lambda_1^{(n)}$	$(\lambda_1)_{j\in[n]} \in \mathcal{P}_n$	Section 2.2.1
$\alpha(\lambda, \mu; \mathfrak{o})$	number of subgroups of isomorphism type μ of a finite abelian p-group of isomorphism type λ	Section 2.2.1
$\lvert\lambda\rvert$	$\sum_{i=1}^n \lambda_i$	Section 2.2.2
$\overline{F}$	interior of a monoid F in $\mathbb{N}_0^m$	Section 2.4
$X(\mathbf{Z})$	generating function of $X \subseteq \mathbb{N}_0^m$	Section 2.5
$D_{\overline{F}}$	specific finite subset of a monoid F in $\mathbb{N}_0^m$	(2.6)
$\mathcal{W}_d$	set of relevant pairs (I, σ) with $I \subseteq [d-1]$ and $\sigma \in S_{2d'}$	Definition 3.11
$G_{I,\sigma}$	specific subset of $\mathbb{N}_0^{d+d'}$ for each $(I,\sigma) \in \mathcal{W}_d$	Definition 3.13
$\mathbf{X}$	tuple of indeterminates $(X_i)_{i\in[d]}$	Section 3.2
$\mathbf{Y}$	tuple of indeterminates $(Y_j)_{j\in[d']}$	Section 3.2
$H_{I,J}$	specific subset of $\mathbb{N}_0^{d+d'}$ for each $I \subseteq [d-1]$ and $J \subseteq [d'-1]$	Definition 3.30
w_σ	Dyck word associated with $\sigma \in S_{2d'}$	Definition 3.37
μ_λ	integer partition $(\mu_j)_{j\in[d']} \in \mathcal{P}_{d'}$ such that the multisets $\{\mu_j \mid j \in [d']\}$ and $\{\lambda_i + \lambda_{i'} \mid i < i' \in [d]\}$ coincide	Definition 4.1
$\mathrm{GMC}_{I,\sigma}$	product of Gaussian multinomial coefficients associated with $(I,\sigma) \in \mathcal{W}_d$	Definition 4.19
χ_σ	numerical data map	Definition 4.21
$\zeta^w_{\mathfrak{f}_{2,d}}(q,t)$	bivariate rational expression such that the evaluation $\zeta^w_{\mathfrak{f}_{2,d}}(q_\mathfrak{o}, q_\mathfrak{o}^{-s})$ equals $\zeta^w_{\mathfrak{f}_{2,d}(\mathfrak{o})}(s)$ for all $\mathfrak{o}$	Section 5.1
$\zeta^{\text{n.o.}}_{\mathfrak{f}_{2,d}}(q,t)$	bivariate rational expression such that the evaluation $\zeta^{\text{n.o.}}_{\mathfrak{f}_{2,d}}(q_\mathfrak{o}, q_\mathfrak{o}^{-s})$ equals $\zeta^{\text{n.o.}}_{\mathfrak{f}_{2,d}(\mathfrak{o})}(s)$ for all $\mathfrak{o}$	Section 5.1
$\chi_{\text{n.o.}}$	no-overlap numerical data map	Section 5.2
$\mathrm{MC}_{I,\sigma}$	product of multinomial coefficients associated with (I,σ)	Definition 6.1
$a_\sigma(\alpha)$ $b_\sigma(\alpha)$	non-negative integers for each $\sigma \in \mathcal{S}_{2d'}$ and $\alpha \in \mathbb{N}_0^{m_\sigma}$ that are closely related to the numerical data map χ_σ	Definition 6.2
$U_{I,\sigma,\max}$	set of $u \in U_{I,\sigma}$ such that $\dim K_u = D$	Definition 6.4
c_d	specific positive rational number depending only on d	Definition 6.4
$\zeta^{\text{red}}_{\mathfrak{f}_{2,d}}(t)$	reduced subalgebra zeta function of $\mathfrak{f}_{2,d}$	Section 6.2
χ_{red}	reduced numerical data map	Definition 6.6
$\zeta^{\text{top}}_{\mathfrak{f}_{2,d}}(s)$	topological subalgebra zeta function of $\mathfrak{f}_{2,d}$	Section 6.3

Table 1.1. Notation.

out to us how to use `Zeta` [11] to efficiently write large sums of rational functions of a specific form on a common denominator. This work was partly funded by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) — SFB-TRR 358/1 2023 — 491392403.

2. Preliminaries

2.1. **Gaussian binomial and multinomial coefficients.** We start by recalling Gaussian binomial and multinomial coefficients.

Definition 2.1. Let $k, n \in \mathbb{N}_0$ with $k \leqslant n$. The *Gaussian binomial coefficient* or *q-binomial coefficient* $\binom{n}{k}_q$ is the following polynomial in q:

$$\binom{n}{k}_q := \frac{(1-q^n)(1-q^{n-1})\cdots(1-q^{n-k+1})}{(1-q^k)(1-q^{k-1})\cdots(1-q^1)}.$$

Definition 2.2. Let $n \in \mathbb{N}_0$ and $J = \{j_i \mid i \in [r]\} \subseteq [n-1]$ with $j_1 \leqslant \cdots \leqslant j_r$. The *Gaussian multinomial coefficient* $\binom{n}{J}_q$ is the polynomial in q defined as

$$\binom{n}{J}_q := \binom{n}{j_1}_q \binom{n-j_1}{j_2-j_1}_q \cdots \binom{n-j_{r-1}}{j_r-j_{r-1}}_q.$$

We write S_n for the symmetric group of degree n, a Coxeter group with Coxeter generators $s_1, \ldots, s_{n-1}$. For $\sigma \in S_n$, we write $\ell(\sigma)$ for the Coxeter length of σ and $\mathrm{Des}(\sigma) = \{i \in [n-1] \mid \ell(\sigma s_i) < \ell(\sigma)\}$ for its (right) descent set. The unique ℓ-longest element in S_n is denoted σ_0, with $\ell(\sigma_0) = \binom{n}{2}$. The identities

$$\ell(\sigma\sigma_0) = \ell(\sigma_0) - \ell(\sigma), \quad \mathrm{Des}(\sigma\sigma_0) = [n-1]\setminus \mathrm{Des}(\sigma),$$

and

$$\binom{n}{J}_q = \sum_{\sigma \in S_n,\, \mathrm{Des}(\sigma) \subseteq J} q^{\ell(\sigma)} \tag{2.1}$$

for $J \subseteq [n-1]$ are well-known. We represent permutations $\sigma \in S_n$ by their *one-line notation*, i.e. the word $\sigma(1)\sigma(2)\ldots\sigma(n)$ in the letters $[n]$.

2.2. **Counting submodules of $\mathfrak{o}$-modules.** We recall some well-known facts about the enumeration of submodules of finitely generated $\mathfrak{o}$-modules, where $\mathfrak{o}$ is a cDVR. We consider torsion modules in Section 2.2.1 and torsion-free modules in Section 2.2.2.

2.2.1. *Finite $\mathfrak{o}$-modules.* Let $\mathcal{P}_n \subset \mathbb{N}_0^n$ be the set of integer partitions of at most n (non-zero) parts, i.e. the set of tuples $\lambda = (\lambda_j)_{j \in [n]}$ with $\lambda_i \in \mathbb{N}_0$ for $i \in [n]$ and $\lambda_i \geqslant \lambda_{i+1}$ for $i \in [n-1]$. By convention, $\lambda_{n+1} = 0$ for $\lambda \in \mathcal{P}_n$. We call a finite $\mathfrak{o}$-module *of isomorphism type* λ if it is isomorphic to the product $C_{\mathfrak{o},\lambda} := \mathfrak{o}/\mathfrak{p}^{\lambda_1} \times \cdots \times \mathfrak{o}/\mathfrak{p}^{\lambda_n}$ of finite cyclic $\mathfrak{o}$-modules.

Let $\lambda, \mu \in \mathcal{P}_n$. We write $\mu \leqslant \lambda$ if $\mu_i \leqslant \lambda_i$ for every $i \in [n]$. Let $\alpha(\lambda, \mu; \mathfrak{o})$ be the number of submodules of $C_{\mathfrak{o},\lambda}$ of isomorphism type μ. The following formula for $\alpha(\lambda, \mu; \mathfrak{o})$ was recorded (at least in the case $\mathfrak{o} = \mathbb{Z}_p$, i.e. for finite abelian p-groups) in [4]. We denote by λ' and μ' the conjugate partitions of λ and μ, respectively.

Proposition 2.3 ([4])**.** *Let $\mu \leqslant \lambda$ be partitions, with conjugate partitions $\mu' \leqslant \lambda'$. Then*

$$\alpha(\lambda, \mu; \mathfrak{o}) = \prod_{k \geqslant 1} q_{\mathfrak{o}}^{\mu'_k(\lambda'_k - \mu'_k)} \binom{\lambda'_k - \mu'_{k+1}}{\mu'_k - \mu'_{k+1}}_{q_{\mathfrak{o}}^{-1}}. \tag{2.2}$$

For later use, we obtain a slightly different expression for $\alpha(\lambda, \mu; \mathfrak{o})$.

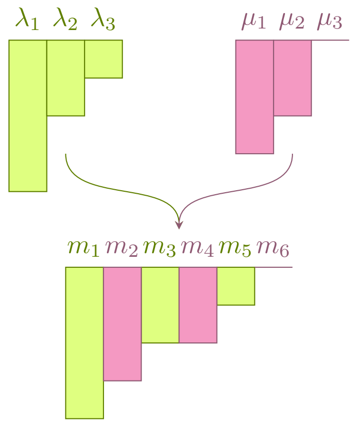

FIGURE 2.1. Illustration of Example 2.5.

Definition 2.4. Let $\mu \leqslant \lambda \in \mathcal{P}_n$. Let $m_j \in \mathbb{N}_0$ for $j \in [2n]$ be such that multisets $\lambda \cup \mu$ and $\{m_j\}_{j\in[2n]}$ are equal and $m_1 \geqslant m_2 \geqslant \cdots \geqslant m_{2n}$. Let $L_0 = M_0 = 0$ and

$$L_j = \#\{i \in [n] \mid \lambda_i \geqslant m_j\},$$
$$M_j = \#\{i \in [n] \mid \mu_i \geqslant m_j\},$$

for $j \in [2n]$.

We note that similar but different integers L_j and M_j are defined in [13, (2.13)].

Example 2.5. For $\lambda = (4, 2, 1)$ and $\mu = (3, 2, 0)$ in $\mathcal{P}_3$, we find that $(m_j)_{j\in[6]} = (4, 3, 2, 2, 1, 0)$, $(L_j)_{j\in\{0,\dots,6\}} = (0, 1, 1, 2, 2, 3, 3)$, and $(M_j)_{j\in\{0,\dots,6\}} = (0, 0, 1, 2, 2, 2, 3)$. See Figure 2.1 for an illustration of this example.

The following lemma resembles [13, Lemmas 2.16 and 2.17].

Lemma 2.6. *Let $\lambda, \mu \in \mathcal{P}_n$ with $\mu \leqslant \lambda$. Then*

$$\alpha(\lambda, \mu; \mathfrak{o}) = \prod_{j=1}^{2n} \binom{L_j - M_{j-1}}{M_j - M_{j-1}}_{q_\mathfrak{o}^{-1}} q_\mathfrak{o}^{M_j(L_j - M_j)(m_j - m_{j+1})}. \tag{2.3}$$

Proof. The product in (2.2) is indexed by integers k. Suppose that $j \in [2n]$ and $k \in [\lambda_1]$ are such that $m_j \geqslant k > m_{j+1}$. Then $\lambda'_k = L_j$ and $\mu'_k = M_j$. Hence (2.2) reads

$$\alpha(\lambda, \mu; \mathfrak{o}) = \prod_{j=1}^{2n} \prod_{k=m_{j+1}+1}^{m_j} q_\mathfrak{o}^{M_j(L_j - M_j)} \binom{L_j - \mu'_{k+1}}{M_j - \mu'_{k+1}}_{q_\mathfrak{o}^{-1}}.$$

Now μ'_{k+1} is equal to M_j if $m_j > k > m_{j+1}$ and equal to M_i if $k = m_j$ and $i = \max(\{i \in [2n] \mid M_i < M_j\})$. If $\mu'_{k+1} = M_j$, then $\binom{L_j - \mu'_{k+1}}{M_j - \mu'_{k+1}}_{q_\mathfrak{o}^{-1}} = 1$. Removing these factors from the product we obtain

$$\alpha(\lambda, \mu; \mathfrak{o}) = \prod_{j=1}^{2n} \binom{L_j - M_{j-1}}{M_j - M_{j-1}}_{q_\mathfrak{o}^{-1}} \prod_{k=m_{j+1}+1}^{m_j} q_\mathfrak{o}^{M_j(L_j - M_j)}$$
$$= \prod_{j=1}^{2n} \binom{L_j - M_{j-1}}{M_j - M_{j-1}}_{q_\mathfrak{o}^{-1}} q_\mathfrak{o}^{M_j(L_j - M_j)(m_j - m_{j+1})}. \qquad \square$$

Given $\lambda_1 \in \mathbb{N}_0$, we write $\lambda_1^{(n)}$ for $(\lambda_1)_{j\in[n]} \in \mathcal{P}_n$. The following is obvious.

Corollary 2.7. *Let $\lambda \in \mathcal{P}_n$ and set $I := \{i \in [n-1] \mid \lambda_i > \lambda_{i+1}\}$. Then*

$$\alpha\left(\lambda_1^{(n)}, \lambda; \mathfrak{o}\right) = \binom{n}{I}_{q_{\mathfrak{o}}^{-1}} \prod_{j=1}^{n} q_{\mathfrak{o}}^{j(n-j)(\lambda_j-\lambda_{j+1})}.$$

2.2.2. *Free $\mathfrak{o}$-modules.* Let π be a uniformiser of $\mathfrak{o}$, i.e. a generator of $\mathfrak{p}$.

Definition 2.8. Let Λ be a submodule of $\mathfrak{o}^n$ of finite index. Let $\{\pi^{\lambda_j}\}_{j\in[n]}$ with $\lambda_1 \geqslant \cdots \geqslant \lambda_n$ be the multiset of elementary divisors of $\mathfrak{o}^n/\Lambda$. The partition $\lambda = (\lambda_j)_{j\in[n]} \in \mathcal{P}_n$ is the *elementary divisor type* of Λ, written $\varepsilon(\Lambda) = \lambda$.

We note that Corollary 2.7 yields the number of submodules of $\mathfrak{o}^n$ of elementary divisor type λ. This proves the following proposition, which counts submodules of fixed elementary divisor type. Given $\lambda = (\lambda_j)_{j\in[n]} \in \mathcal{P}_n$, we set $|\lambda| = \sum_{i=1}^n \lambda_i$.

Proposition 2.9 ([5, Section 4.2])**.** *Given a partition $\lambda = (\lambda_j)_{j\in[n]} \in \mathcal{P}_n$,*

$$\sum_{\substack{\Lambda \leqslant \mathfrak{o}^n \\ \varepsilon(\Lambda)=\lambda}} |\mathfrak{o}^n : \Lambda|^{-s} = \alpha(\lambda_1^{(n)}, \lambda; \mathfrak{o}) q_{\mathfrak{o}}^{-s|\lambda|}.$$

The following proposition counts submodules containing a given submodule.

Proposition 2.10 ([5, Section 4.3])**.** *Let $M \leqslant \mathfrak{o}^n$ be a submodule with elementary divisor type $\varepsilon(M) = \mu$. Then*

$$\sum_{\substack{\Lambda \leqslant \mathfrak{o}^n \\ \Lambda \geqslant M}} |\mathfrak{o}^n : \Lambda|^{-s} = \sum_{\substack{\nu \in \mathcal{P}_n \\ \nu \leqslant \mu}} \alpha(\mu, \nu; \mathfrak{o})\, q_{\mathfrak{o}}^{-s|\nu|}.$$

Proof. Observe that $\mathfrak{o}^n/M \cong C_{\mathfrak{o},\mu}$ and $|\mathfrak{o}^n : \Lambda| = q_{\mathfrak{o}}^{|\nu|}$ if $\varepsilon(\Lambda) = \nu$. □

2.3. **Convex polyhedral cones in $\mathbb{Q}^m$.** We recall some general nomenclature and results for convex polyhedral cones. We largely follow [14, p. 477].

The *dimension* $\dim A$ of a subset $A \subseteq \mathbb{Q}^m$ is the dimension of the subspace of $\mathbb{Q}^m$ generated by A. A *cone* in $\mathbb{Q}^m$ is a subset $\mathcal{C} \subseteq \mathbb{Q}^m$ that is closed under addition and scaling by non-negative rational numbers. The convex cone generated by $A \subseteq \mathbb{Q}^m$ is

$$\mathcal{C}_A := \{a_1x_1 + \cdots + a_tx_t \mid x_1, \ldots, x_t \in A, a_1, \ldots, a_t \in \mathbb{Q}_{\geqslant 0}\}.$$

A *linear half-space* of $\mathbb{Q}^m$ is a subset of $\mathbb{Q}^m$ of the form $\mathcal{H} = \{v \in \mathbb{Q}^m \mid w \cdot v \geqslant 0\}$ for a vector $w \in \mathbb{Q}^m \backslash \{0\}$. A *convex polyhedral cone* $\mathcal{C}$ is the intersection of finitely many half-spaces. It is *pointed* if it does not contain a line.

Let $\mathcal{C}$ be a convex polyhedral cone in $\mathbb{Q}^m$. A *supporting hyperplane* for $\mathcal{C}$ is a hyperplane $\mathcal{H}$ that divides $\mathbb{Q}^m$ into two linear half-spaces $\mathcal{H}^+$ and $\mathcal{H}^-$ such that $\mathcal{C} \subseteq \mathcal{H}^+$ or $\mathcal{C} \subseteq \mathcal{H}^-$. A *face* $\mathcal{F}$ of $\mathcal{C}$ is either an intersection $\mathcal{C} \cap \mathcal{H}$ of $\mathcal{C}$ with a supporting hyperplane $\mathcal{H}$ or equal to $\mathcal{C}$. Faces of dimension one are called *extreme rays* and faces of dimension $m-1$ are called *facets*. The convex polyhedral cone $\mathcal{C}$ is *simplicial* if it has exactly $\dim \mathcal{C}$ extreme rays.

For $x \in \mathbb{Q}^m$ and $\varepsilon > 0$, let $N_\varepsilon(x)$ be the closed ball or radius ε around x. Let A be any subset of $\mathbb{Q}^m$ and $\mathrm{aff}(A)$ the affine hull of A. The *relative interior* $\mathrm{relint}(A)$ of A is the set of points $a \in A$ such that there is an $\varepsilon > 0$ with $N_\varepsilon(x) \cap \mathrm{aff}(A)$ contained in A. If A is a convex polyhedral cone, then the relative interior of A is the set of points in A that are not contained in any face of A of lower dimension than A.

Definition 2.11 ([14, p. 477]). Let $\mathcal{C}$ be a convex polyhedral cone in $\mathbb{Q}^m$. A *triangulation* of $\mathcal{C}$ is a finite family $\Gamma = \{\mathcal{K}_u\}_{u\in U}$ of simplicial polyhedral cones $\mathcal{K}_u$ such that the following hold:

- $\mathcal{C} = \bigcup_{u\in U} \mathcal{K}_u$,

- for each $\mathcal{K}_u \in \Gamma$, every face of $\mathcal{K}_u$ is an element of Γ, and
- for every $\mathcal{K}_u, \mathcal{K}_v \in \Gamma$, the intersection $\mathcal{K}_u \cap \mathcal{K}_v$ is a common face of $\mathcal{K}_u$ and $\mathcal{K}_v$.

An element of Γ is called a *face* of Γ.

Remark 2.12. If $\Gamma = \{\mathcal{K}_u\}_{u \in U}$ is a triangulation of $\mathcal{C}$, then $\mathcal{C} = \bigcup_{u \in U} \operatorname{relint}(\mathcal{K}_u)$ and this union is disjoint.

Proposition 2.13 ([14, Lem. 4.5.1])**.** *Every pointed polyhedral cone $\mathcal{C}$ has a triangulation whose one-dimensional faces are the extreme rays of $\mathcal{C}$.*

Let P be a poset with partial order relation $\leqslant_P$. An *interval* in P is a subset of P of the form $[x, y] = \{z \in P \mid x \leqslant_P z \leqslant_P y\}$ for some $x \leqslant_P y \in P$. An interval is *non-trivial* if $x <_P y$. The element x *covers* y in P if $[y, x] = \{x, y\}$. The poset P is *graded* if it is endowed with a rank function $\operatorname{rk} : P \to \mathbb{N}_0$, i.e. a function satisfying $\operatorname{rk}(x) > \operatorname{rk}(y)$ if $x >_P y$ in P and $\operatorname{rk}(x) = \operatorname{rk}(y) + 1$ if x covers y. A graded poset $\mathcal{P}$ is *Eulerian* if, in any non-trivial interval, the number of elements of even rank and the number of elements odd rank coincide.

Let $\mathcal{C}$ be a convex polyhedral cone in $\mathbb{Q}^m$. The *lattice of faces* $L(\mathcal{C})$ is the poset consisting of the faces of $\mathcal{C}$ ordered by inclusion. The following is well known and is a consequence of [3, Cor. 3.5.4] and [3, Cor. 3.3.3].

Proposition 2.14. *The lattice of faces $L(\mathcal{C})$ of $\mathcal{C}$ is Eulerian. If $\mathcal{C}$ is pointed, then the rank of a face $\mathcal{F}$ is* $\dim(\mathcal{F})$.

2.4. **Monoids in $\mathbb{N}_0^m$.** We discuss monoids in $\mathbb{N}_0^m$, in particular those that are associated with systems of homogeneous linear equations with integer coefficients. We largely follow [14, Sec. 4.5].

A *monoid* F in $\mathbb{N}_0^m$ is a subset of $\mathbb{N}_0^m$ that contains zero and is closed under addition. The *interior* of F, denoted by $\overline{F}$, is the set of points in F that lie in the relative interior of $\mathcal{C}_F$. The *completely fundamental* elements $\mathrm{CF}(F)$ of F are the elements $\beta \in F$ such that if $n \in \mathbb{N}$ and $\alpha, \alpha' \in F$ are such that $n\beta = \alpha + \alpha'$, then $\alpha = i\beta$ and $\alpha' = (n - i)\beta$ for some $i \in \mathbb{N}_0$ with $i \leqslant n$. A system of r homogeneous linear equations with integer coefficients in m variables $\alpha_1, \ldots, \alpha_m$ can always be written as $\Phi(\alpha_1, \ldots, \alpha_m) = 0$ for an $r \times m$ matrix Φ over $\mathbb{Z}$. The set of solutions in $\mathbb{N}_0^m$ of this system,

$$E = E_\Phi := \{\alpha \in \mathbb{N}_0^m \mid \Phi\alpha = 0\}, \tag{2.4}$$

is a monoid in $\mathbb{N}^m$.

Remark 2.15. The convex cone $\mathcal{C}_E$ generated by E is a pointed convex polyhedral cone. The completely fundamental elements of E each generate an extreme ray of $\mathcal{C}_E$ and vice versa.

Remark 2.16. We may assume that the rank of Φ is r by deleting dependent rows of Φ. If $E \cap \mathbb{N}^m = \varnothing$, then there must be an $i \in [m]$ such that the i-th entry of α is 0 for every $\alpha \in \mathcal{C}$. In this case, we can just ignore the i-th coordinate. In general, we assume that no coordinates are redundant and that $E \cap \mathbb{N}^m \neq \varnothing$. It then follows that the interior $\overline{E}$ is the set $E \cap \mathbb{N}^m$ of positive points in E.

Remark 2.17. Through slack variables, (2.4) can be used to study monoids defined by linear inequalities as well. Concretely, let $\Phi \in \mathrm{Mat}_{k \times m}(\mathbb{Z})$ and consider the monoid $\mathcal{S} = \{v \in \mathbb{N}_0^m \mid \Phi v \geqslant 0\}$. The points in $\mathcal{S}$ are in bijection with the points in

$$\left\{v \in \mathbb{N}_0^m, \gamma \in \mathbb{N}_0^k \mid \Phi v - \gamma = 0\right\},$$

where γ is a tuple of k (slack) variables.

A monoid F in $\mathbb{N}_0^m$ is *simplicial* if there exist $\mathbb{Q}$-linearly independent tuples $\alpha_1, \dots, \alpha_t \in F$ called *quasigenerators* of F such that

$$F = \{\gamma \in \mathbb{N}^m \mid \exists n \in \mathbb{N}, \exists a_1, \dots, a_t \in \mathbb{N}_0 : n\gamma = a_1\alpha_1 + \cdots + a_t\alpha_t\}.$$

A monoid F is simplicial if and only if $\mathcal{C}_F$ is a simplicial polyhedral cone. In that case, the completely fundamental elements $\mathrm{CF}(F)$ are quasigenerators of F. The interior $\overline{F}$ of a simplicial monoid F can be characterised by

$$\overline{F} = \{\gamma \in \mathbb{N}^m \mid \exists n \in \mathbb{N}, \exists a_1, \dots, a_t \in \mathbb{N} : n\gamma = a_1\alpha_1 + \cdots + a_t\alpha_t\}.$$

The *support* of a tuple $x = (x_1, \dots, x_m) \in \mathbb{Q}_0^m$ is the set $\mathrm{supp}(x) := \{i \in [m] \mid x_i \neq 0\}$. The *support* of a set $V \subseteq \mathbb{Q}^m$ is $\mathrm{supp}(V) := \bigcup_{v \in V} \mathrm{supp}(v)$. Suppose that $E = E_\Phi$ for some $\Phi \in \mathrm{Mat}_{r\times m}(\mathbb{Z})$. The *lattice of supports* $L(E)$ of E is the set $\{\mathrm{supp}(\alpha) \mid \alpha \in E\}$ of supports of tuples in E, ordered by inclusion. The next result identifies the posets $L(\mathcal{C}_E)$ and $L(E)$.

Theorem 2.18 ([14, p. 479]). *The map*

$$L(\mathcal{C}_E) \to L(E) : \mathcal{F} \mapsto \mathrm{supp}(\mathcal{F})$$

is a poset isomorphism.

Because of Remark 2.16, we may assume that $L(E)$ has $[m]$ as greatest element.

2.5. **Generating functions of subsets of $\mathbb{N}_0^m$.** We discuss a generating function associated with subsets of $\mathbb{N}_0^m$, in particular monoids and their interiors. We largely follow [14, Sec. 4.5].

For a subset $X \subseteq \mathbb{N}_0^m$, define the generating function

$$X(\mathbf{Z}) := \sum_{\boldsymbol{\alpha} \in X} \mathbf{Z}^{\boldsymbol{\alpha}} \in \mathbb{Q}[\![\mathbf{Z}]\!], \tag{2.5}$$

in the indeterminates $\mathbf{Z} = (Z_j)_{j\in[m]}$, where $\mathbf{Z}^{\boldsymbol{\alpha}} = \prod_{j\in[m]} Z_j^{\alpha_j}$ for $\boldsymbol{\alpha} = (\alpha_j)_{j\in[m]} \in \mathbb{N}_0^m$. The sets X whose generating functions we consider are often monoids F or $E = E_\Phi$, or the interior $\overline{F}$ or $\overline{E}$ of such monoids.

Consider a monoid F in $\mathbb{N}_0^m$ that is simplicial with quasigenerators $\alpha_1, \dots, \alpha_t$. Define the following finite subsets of F which depend on the choice of quasigenerators:

$$D_F := \{x \in F \mid x = a_1\alpha_1 + \ldots + a_t\alpha_t,\ 0 \leqslant a_i < 1\},$$
$$D_{\overline{F}} := \{x \in F \mid x = a_1\alpha_1 + \ldots + a_t\alpha_t,\ 0 < a_i \leqslant 1\}.$$

Theorem 2.19 ([14, Cor. 4.5.8]). *Let $F \subseteq \mathbb{N}_0^m$ be a simplicial monoid with quasigenerators $\alpha_1, \dots, \alpha_t$. The generating functions $F(\mathbf{Z})$ and $\overline{F}(\mathbf{Z})$ are rational and given by:*

$$F(\mathbf{Z}) = \frac{\sum_{\beta \in D_F} \mathbf{Z}^\beta}{\prod_{i=1}^t (1 - \mathbf{Z}^{\alpha_i})},$$
$$\overline{F}(\mathbf{Z}) = \frac{\sum_{\beta \in D_{\overline{F}}} \mathbf{Z}^\beta}{\prod_{i=1}^t (1 - \mathbf{Z}^{\alpha_i})}. \tag{2.6}$$

For monoids E of the form (2.4), we have the following result.

Theorem 2.20 ([14, Theorem 4.5.11]). *Let $E = E_\Phi$ for some $\Phi \in \mathrm{Mat}_{r\times m}(\mathbb{Z})$. The generating functions $E(\mathbf{Z})$ and $\overline{E}(\mathbf{Z})$ are rational and when written in lowest terms, they both have denominator*

$$\prod_{\beta \in \mathrm{CF}(E)} \left(1 - \mathbf{Z}^\beta\right).$$

The following two theorems are reciprocity results for $X(\mathbf{Z})$, for simplicial monoids and monoids of the form (2.4) respectively. Let $\mathbf{Z}^{-1} = (Z_j^{-1})_{j\in[m]}$.

Theorem 2.21 ([14, Lemma 4.5.13])**.** *Let $F \subseteq \mathbb{N}^m$ be a simplicial monoid of dimension n. Then*

$$\overline{F}(\mathbf{Z}^{-1}) = (-1)^n F(\mathbf{Z}).$$

Theorem 2.22 ([14, Theorem 4.5.14])**.** *Let $E = E_\Phi$ for some $\Phi \in \mathrm{Mat}_{r\times m}(\mathbb{Z})$ and n be the dimension of E. Then*

$$\overline{E}(\mathbf{Z}^{-1}) = (-1)^n E(\mathbf{Z}).$$

2.6. **The submonoids $F_{E,A}$ and subsets $\overline{F}_{E,A}$ of $E = E_\Phi$.** Let $E = E_\Phi$ for some $\Phi \in \mathrm{Mat}_{r\times m}(\mathbb{Z})$. We define submonoids $F_{E,A}$ and subsets $\overline{F}_{E,A}$ of E where A is an element of the lattice of supports $L(E)$ of E. We formulate some reciprocity results for their generating functions.

For $A \in L(E)$, define

$$F_{E,A} := \{\alpha \in E \mid \mathrm{supp}(\alpha) \subseteq A\},$$
$$\overline{F}_{E,A} := \{\alpha \in E \mid \mathrm{supp}(\alpha) = A\}.$$

The first set, $F_{E,A}$, is a submonoid of E. The convex cone $\mathcal{C}_{F_{E,A}}$ is the unique (cf. Theorem 2.18) face of $\mathcal{C}_E$ whose support is A. In other words, $\mathcal{C}_{F_{E,A}}$ is the inverse image of A under the map in (2.18). The second set $\overline{F}_{E,A}$ is a subset of E and is the interior of $F_{E,A}$. Clearly, $E = F_{E,[m]}$ and

$$F_{E,A} = \bigcup_{B\in L(E), B\subseteq A} \overline{F}_{E,B}, \tag{2.7}$$

where the union is disjoint.

Remark 2.23. If $\alpha \in F_{E,A}$, then the i-th coordinate of α is zero for all $i \in [m]\backslash A$. Therefore, the coordinates $[m]\backslash A$ of $F_{E,A}$ may be discarded. That way $F_{E,A} = E_{\Phi_A}$, where Φ_A is the matrix found by removing the columns $[m]\backslash A$ from Φ and deleting resulting dependent rows.

Remark 2.24. For every $A \in L(E)$, let $\mathbf{Z}_A$ be the tuple $(Z_j)_{j\in[m]}$ where Z_j is an indeterminate for $j \in A$ and $Z_j = 0$ for $j \in [m]\backslash A$. Then $F_{E,A}(\mathbf{Z}) = E(\mathbf{Z}_A)$.

The following proposition is a reciprocity result for the submonoids $F_{E,A} \subseteq E$ and subsets $\overline{F}_{E,A} \subseteq E$.

Proposition 2.25. *Let $E = E_\Phi$ for some $\Phi \in \mathrm{Mat}_{r\times m}(\mathbb{Z})$. For all $A \in L(E)$,*

$$\overline{F}_{E,A}(\mathbf{Z}^{-1}) = (-1)^{\dim F_{E,A}} F_{E,A}(\mathbf{Z}) = (-1)^{\dim \overline{F}_{E,A}} \sum_{B\in L(E), B\subseteq A} \overline{F}_{E,B}(\mathbf{Z}). \tag{2.8}$$

Proof. The first equality is an application of Theorem 2.22 to $F_{E,A}$, which is applicable because of Remark 2.23. Recall that $\overline{F}_{E,A}$ is the interior of $F_{E,A}$. Using (2.7), the second equality follows. □

Corollary 2.27 states an alternative reciprocity result for $\overline{F}_{E,A}$ that is analogous to [18, Lemma 2.17]. To prove it, we need the following lemma.

Lemma 2.26. *Let $E = E_\Phi$ for some $\Phi \in \mathrm{Mat}_{r\times m}(\mathbb{Z})$. For all $A, C \in L(E)$ with $A \subseteq C$,*

$$\sum_{B\in L(E), A\subseteq B\subseteq C} (-1)^{\dim F_{E,B}} = \begin{cases} (-1)^{\dim F_{E,A}} & \text{if } A = C, \\ 0 & \text{otherwise.} \end{cases} \tag{2.9}$$

Proof. If $A = C$, then the summation in (2.9) has exactly one summand, namely $(-1)^{\dim F_{E,A}}$. In general, the set $\{B \in L(E) \mid A \subseteq B \subseteq C\}$ is an interval in $L(E)$. If $A \neq C$, then it is a non-trivial interval. Combining Proposition 2.14 with Theorem 2.18, we find that $L(E)$ is an Eulerian poset where the rank of $B \in L(E)$ is $\dim F_{E,B}$. Recall that Eulerian means that in every non-trivial interval, the number of elements of even rank and the number of elements of odd rank coincide. Therefore, the summation (2.9) completely cancels out if $A \neq C$. ☐

Corollary 2.27. *Let $E = E_\Phi$ for some $\Phi \in \mathrm{Mat}_{r\times m}(\mathbb{Z})$ and n be the dimension of E. Let $A \subseteq \mathrm{supp}(E)$ be such that $A \in L(E)$ or $\mathrm{supp}(E)\backslash A \in L(E)$. Then*

$$\sum_{B\in L(E),B\supseteq A} \overline{F}_{E,B}(\mathbf{Z}^{-1}) = (-1)^n \sum_{C\in L(E),C\supseteq \mathrm{supp}(E)\backslash A} \overline{F}_{E,C}(\mathbf{Z}). \tag{2.10}$$

Proof (adapted from [18, Lemma 2.17]*).* Using (2.8), we find that

$$\begin{aligned}\sum_{B\in L(E),B\supseteq A} \overline{F}_{E,B}(\mathbf{Z}^{-1}) &= \sum_{B\in L(E),B\supseteq A} (-1)^{\dim \overline{F}_{E,B}} \sum_{C\in L(E),C\subseteq B} \overline{F}_{E,C}(\mathbf{Z}) \\ &= \sum_{C\in L(E)} \left(\sum_{B\in L(E),B\supseteq C\cup A} (-1)^{\dim F_{E,B}} \right) \overline{F}_{E,C}(\mathbf{Z}), \end{aligned} \tag{2.11}$$

where we used that $\dim \overline{F}_{E,B} = \dim F_{E,B}$. If $A \in L(E)$, then also $A \cup C \in L(E)$ for $C \in L(E)$. Therefore it follows from Lemma 2.26, that the expression between brackets in (2.11) is $(-1)^d$ when $C \cup A = \mathrm{supp}(E)$ or equivalently $C \supseteq \mathrm{supp}(E)\backslash A$, and zero otherwise. Thus (2.10) holds if $A \in L(E)$. Notice that the roles of A and $\mathrm{supp}(E)\backslash A$ in (2.10) are symmetric, so (2.10) also holds if $\mathrm{supp}(E)\backslash A \in L(E)$. ☐

2.7. **The subsets $I_{E,A,C}$ of $E = E_\Phi$.** Let $E = E_\Phi$ for some $\Phi \in \mathrm{Mat}_{r\times m}(\mathbb{Z})$. We define subsets $I_{E,A,C}$ of E for all $A, C \in L(E)$ with $A \subseteq C$. We show a reciprocity result for their generating functions and specify a decomposition as a disjoint union of interiors of simplicial monoids.

For all $A, C \subseteq [m]$ with $A \subseteq C$, define

$$I_{E,A,C} := \{\alpha \in E \mid A \subseteq \mathrm{supp}(\alpha) \subseteq C\}.$$

In other words, the elements in $I_{E,A,C}$ are the elements of E that have positive entries in the coordinates indexed by elements in A, non-negative entries in the coordinates indexed by elements in $C\backslash A$ and zeroes elsewhere. Obviously,

$$I_{E,A,C} = \bigcup_{B\in L(E),A\subseteq B\subseteq C} \overline{F}_{E,B}, \tag{2.12}$$

where the union is disjoint.

We formulate a reciprocity result for $I_{E,A,C}$ when $A, C \in L(E)$ and $C\backslash A \in L(E)$.

Proposition 2.28. *Let $E = E_\Phi$ for some $\Phi \in \mathrm{Mat}_{r\times m}(\mathbb{Z})$. Suppose that $A, C \in L(E)$ with $A \subseteq C$ and $C\backslash A \in L(E)$. Then*

$$I_{E,A,C}(\mathbf{Z}^{-1}) = (-1)^{\dim I_{E,A,C}} I_{E,C\backslash A,C}(\mathbf{Z}). \tag{2.13}$$

Proof. By (2.12),

$$I_{E,A,C}(\mathbf{Z}^{-1}) = \sum_{B\in L(E),A\subseteq B\subseteq C} \overline{F}_{E,B}(\mathbf{Z}^{-1}). \tag{2.14}$$

By (2.8), it follows that

$$I_{E,A,C}(\mathbf{Z}^{-1}) = \sum_{B\in L(E),A\subseteq B\subseteq C} (-1)^{\dim \overline{F}_{E,B}} \sum_{D\in L(E),D\subseteq B} \overline{F}_{E,D}(\mathbf{Z}) \tag{2.15}$$

$$= \sum_{D\in L(E),D\subseteq C} \left(\sum_{B\in L(E),A\cup D\subseteq B\subseteq C} (-1)^{\dim \overline{F}_{E,B}} \right) \overline{F}_{E,D}(\mathbf{Z}).$$

Since, A and D are in $L(E)$, it follows that $A \cup D \in L(E)$. Therefore by Lemma 2.26, the expression between brackets is $(-1)^{\dim \overline{F}_{E,C}}$ when $A \cup D = C$, or equivalently $C\backslash A \subseteq D$, and zero otherwise. Thus we find

$$I_{E,A,C}(\mathbf{Z}^{-1}) = \sum_{D\in L(E),C\backslash A\subseteq D\subseteq C} (-1)^{\dim \overline{F}_{E,C}} \overline{F}_{E,D}(\mathbf{Z}).$$

Since $(-1)^{\dim \overline{F}_{E,C}}$ does not depend on D, it may be pulled out of the summation. Using (2.12), and the fact that $\dim \overline{F}_{E,C} = \dim I_{E,A,C}$, we then find (2.13). □

The following proposition gives a decomposition of $I_{E,A,C}$ as a disjoint union of interiors of simplicial monoids.

Proposition 2.29. *Let $E = E_\Phi$ for some $\Phi \in \mathrm{Mat}_{r\times m}(\mathbb{Z})$ and $A, C \in L(E)$ with $A \subseteq C$. There is a finite family $\Gamma = \{K_u\}_{u\in U}$ of simplicial monoids $K_u \subseteq \mathbb{N}_0^m$ such that*

- *$I_{E,A,C} = \bigcup_{u\in U} \overline{K}_u$ and this union is disjoint,*
- *$\mathrm{CF}(K_u) \subseteq \mathrm{CF}(E)$ for all $u \in U$.*

Proof. (2.12) already writes $I_{E,A,C}$ as a disjoint union of interiors of monoids $F_{E,B}$, but these are not simplicial in general. By Remark 2.23 and Remark 2.15, $\mathcal{C}_{F_{E,B}}$ is a pointed convex polyhedral cone with extreme rays generated by the completely fundamental elements of $F_{E,B}$. Since $\mathcal{C}_{F_{E,B}}$ is a face of $\mathcal{C}_E$, the completely fundamental elements of $F_{E,B}$ are all completely fundamental elements of E. Therefore, by Proposition 2.13, $\mathcal{C}_{F_{E,B}}$ has a triangulation $\Gamma_B = \{\mathcal{K}_u\}_{u\in U_B}$, where each $\mathcal{K}_u$ for $u \in U_B$ is a simplicial polyhedral cone and the one-dimensional faces are generated by a completely fundamental element of E. Let U_B° be the set of $u \in U_B$ such that $\mathcal{K}_u$ is not contained in any of the facets of $\mathcal{C}_{F_{E,B}}$. Equivalently, U_B° is the set of $u \in U_B$ such that $\mathrm{relint}(\mathcal{K}_u)$ is contained in $\mathrm{relint}(\mathcal{C}_{F_{E,B}})$. Set $K_u = \mathcal{K}_u \cap F_{E,B}$ for $u \in U_B^\circ$. Then

$$\overline{F}_{E,B} = \mathrm{relint}(\mathcal{C}_{F_{E,B}}) \cap F_{E,B} = \bigcup_{u\in U_B^\circ} \mathrm{relint}(\mathcal{K}_u) \cap F_{E,B} = \bigcup_{u\in U_B^\circ} \overline{K_u}.$$

Setting $U = \bigcup_{B\in L(E),A\subseteq B\subseteq C} U_B^\circ$, we find using (2.12) that $\bigcup_{u\in U} \overline{K_u} = I_{E,A,C}$ and this union is disjoint because the $\overline{\mathcal{K}_u}$ coming from the same triangulation Γ_B are disjoint, and the union in (2.12) is also disjoint. □

3. The subsets $G_{I,\sigma}$ and $H_{I,J}$ of $\mathbb{N}_0^{d+d'}$

In this section, we define the specific subsets of $\mathbb{N}_0^m$ that are used in the later sections to write down formulas for the considered subalgebra zeta functions. In Sections 3.1 and 3.4, we define monoids $E_\sigma \subseteq \mathbb{N}_0^{m_\sigma}$ and $E_{\text{n.o.}} \subseteq \mathbb{N}_0^{d+d'+1}$. In Sections 3.2 and 3.3, we discuss subsets $G_{I,\sigma}$ of $\mathbb{N}_0^{d+d'}$, which are used to express a formula for $\zeta_{\mathfrak{f}_{2,d}(\mathfrak{o})}(s)$ in Section 4.4 and $\zeta^w_{\mathfrak{f}_{2,d}(\mathfrak{o})}(s)$ in Section 5.1. In Section 3.5, we discuss subsets $H_{I,J}$ of $\mathbb{N}_0^{d+d'}$, which are used to express a formula for $\zeta^{\text{n.o.}}_{\mathfrak{f}_{2,d}(\mathfrak{o})}(s)$ in Section 5.2. We also prove some properties of the associated generating functions that are used to prove properties of $\zeta_{\mathfrak{f}_{2,d}(\mathfrak{o})}(s)$, $\zeta^{w}_{\mathfrak{f}_{2,d}(\mathfrak{o})}(s)$, and $\zeta^{\text{n.o.}}_{\mathfrak{f}_{2,d}(\mathfrak{o})}(s)$ in Section 5. Dyck words and the relation between the $G_{I,\sigma}$ and $H_{I,J}$ are discussed in Section 3.6.

3.1. The monoids $E_\sigma \subseteq \mathbb{N}_0^{m_\sigma}$.

Set $d' := \binom{d}{2}$. We define monoids $E_\sigma \subseteq \mathbb{N}_0^{m_\sigma}$ for certain permutations $\sigma \in S_{2d'}$. This is done by defining a matrix Φ_σ and setting $E_\sigma = E_{\Phi_\sigma}$ as in (2.4).

The permutations $\sigma \in S_{2d'}$ for which we define a monoid E_σ are the following:

Definition 3.1. Let $\mathcal{S}_{2d'}$ be the set of permutations $\sigma \in S_{2d'}$ such that

$$|\{l \in [i] \mid \sigma(l) \leqslant d'\}| \leqslant |\{l \in [i] \mid \sigma(l) > d'\}| \tag{3.1}$$

for all $i \in [2d']$ and if $i < j \in [2d']$ are such that $\sigma(i), \sigma(j) \in [d']$ and $\sigma(i) > \sigma(j)$, then

$$\sigma(i) > \sigma(i+1) > \cdots > \sigma(j-1) > \sigma(j).$$

Example 3.2. Let $d = d' = 3$. Then $123456 \notin \mathcal{S}_{2d'}$ as (3.1) is not satisfied for $i \in [5]$. Also $653421 \notin \mathcal{S}_{2d'}$ because $3 < 5$ are such that $\sigma(3) = 3$, $\sigma(5) = 2 \in [3]$, and $\sigma(3) = 3 > \sigma(5) = 2$, yet $\sigma(3) = 3 \not> \sigma(4) = 4 > \sigma(5) = 2$. However, $451632 \in \mathcal{S}_{2d'}$.

We formalize a way to identify each element of the set $\{(i,j) \in [d]^2 \mid i < j\} \sqcup [d']$ by a unique integer in $[2d']$.

Definition 3.3. Define the bijection

$$\begin{aligned} b : \{(i,j) \in [d]^2 \mid i < j\} \sqcup [d'] &\to [2d'] : \\ (i,j) &\mapsto d' + j - 1 + (i-1)(2d-2-i)/2 \\ j &\mapsto j. \end{aligned}$$

Remark 3.4. The map b respects the lexicographical ordering of the pairs (i,j) with $i < j$.

Example 3.5. If $d = 4$, then b maps

$$\begin{array}{llll} 1 \mapsto 1, & 4 \mapsto 4, & (1,2) \mapsto 7, & (2,3) \mapsto 10, \\ 2 \mapsto 2, & 5 \mapsto 5, & (1,3) \mapsto 8, & (2,4) \mapsto 11, \\ 3 \mapsto 3, & 6 \mapsto 6, & (1,4) \mapsto 9, & (3,4) \mapsto 12. \end{array}$$

Next, we associate a tuple of length $d + d'$ to every element of $[2d']$. For $i \in [d + d']$, let $\delta_i \in \mathbb{N}_0^{d+d'}$ be the tuple whose ith entry is one, while the other entries are zero. Recall that we write $x^{(m)}$ for the tuple $(x)_{j \in [m]}$.

Definition 3.6 (Corresponding tuple). Let $i \in [d']$. The *tuple v_i corresponding to i* is

$$v_i := \sum_{k=d+i}^{d+d'} \delta_k = (0^{(d+i-1)}, 1^{(d'-i+1)}).$$

Let $i \in d' + [d']$ and $b^{-1}(i) = (j,k)$. The *tuple v_i corresponding to i* is

$$v_i := \sum_{l=j}^{d} \delta_l + \sum_{l=k}^{d} \delta_l = (0^{(j-1)}, 1^{(k-j)}, 2^{(d-k+1)}, 0^{(d')}).$$

For $i, j \in [d + d']$, let $v_{i,j}$ be the j-th component of v_i.

Example 3.7. Let $d = 3$ and $i = 4$. Then $b^{-1}(i) = (1,2)$ and $v_4 = (1,2,2,0,0,0)$. Now let $i = 5$. Then $b^{-1}(i) = (1,3)$ and $v_5 = (1,1,2,0,0,0)$.

Using the integers $v_{i,j}$, we define the matrix Φ_σ and monoid E_σ.

Definition 3.8. Let $\sigma \in \mathcal{S}_{2d'}$,

$$R_\sigma := \left\{ i \in [2d'-1] \mid \sigma(i) > d' \text{ or } \sigma(i+1) > d' \right\},$$

$r_\sigma := |R_\sigma|$, and $m_\sigma := d + d' + r_\sigma$. Let $w^\sigma_{i,j} := v_{\sigma(i),j} - v_{\sigma(i+1),j}$ for $i \in [2d'-1]$ and $j \in [d+d']$. Then Φ_σ is the $r_\sigma \times m_\sigma$-matrix, whose row corresponding to $i \in R_\sigma$ has

$$\begin{array}{rl} w^\sigma_{i,j} & \text{in column } j \in [d+d'], \\ -1 & \text{in column } d+d'+i, \\ 0 & \text{in the remaining columns.} \end{array}$$

Let E_σ be the monoid E_{Φ_σ} associated with the matrix $\Phi_{I,\sigma}$ as in (2.4).

Remark 3.9. The matrix Φ_σ in Definition 3.8 is only defined for permutations $\sigma \in \mathcal{S}_{2d'}$ instead of all $\sigma \in S_{2d'}$, partially in order for $E_\sigma \cap \mathbb{N}^m$ to be non-empty as Remark 2.16 requires.

Example 3.10. Let $\sigma = 451632 \in \mathcal{S}_6$. Then $r_\sigma = 4$, $m_\sigma = 10$, and

$$\Phi_\sigma = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 1 & 1 & 2 & -1 & -1 & -1 & 0 & -1 & 0 & 0 \\ 0 & -1 & -2 & 1 & 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 & 0 & -1 & 0 & 0 & 0 & -1 \end{pmatrix}.$$

3.2. **The subsets $G_{I,\sigma} \subseteq \mathbb{N}_0^{d+d'}$.** We define subsets $G_{I,\sigma} \subseteq \mathbb{N}_0^{d+d'}$ for certain pairs (I,σ) with $I \subseteq [d-1]$ and $\sigma \in \mathcal{S}_{2d'}$. These sets $G_{I,\sigma}$ are used in Sections 4.4 and 5.1 to write down formulas for $\zeta_{\mathfrak{f}_{2,d}(\mathfrak{o})}(s)$ and $\zeta^w_{\mathfrak{f}_{2,d}(\mathfrak{o})}(s)$. The pairs (I,σ) for which we define a set $G_{I,\sigma}$ are the following:

Definition 3.11. Let $\mathcal{W}_d$ be the set of pairs (I,σ) with $I \subseteq [d-1]$ and $\sigma \in \mathcal{S}_{2d'}$ such that the following system of inequalities in the variables $r_1, \ldots, r_d$ has non-zero solutions in $\mathbb{N}_0^d$:

$$\begin{cases} r_i > 0 & \text{for } i \in I, \\ r_i = 0 & \text{for } i \in [d'-1]\backslash I, \\ \sum_{k=1}^d (v_{i,k} - v_{j,k}) r_k > 0 & \text{for } i,j \in d + [d'] \text{ with } \sigma^{-1}(i) < \sigma^{-1}(j) \text{ and } i < j, \\ \sum_{k=1}^d (v_{i,k} - v_{j,k}) r_k \geqslant 0 & \text{for } i,j \in d + [d'] \text{ with } \sigma^{-1}(i) < \sigma^{-1}(j) \text{ and } i > j. \end{cases} \tag{3.2}$$

Example 3.12. Let $d = 2$, so $d' = 1$. The set $\mathcal{S}_2$ contains only 21. If $I = \varnothing$, then (3.2) reduces to one equation: $r_1 = 0$, and therefore any $r_2 \in \mathbb{N}$ together with $r_1 = 0$ gives a non-zero solution. If $I = \{1\}$, then (3.2) reduces to one inequality: $r_1 > 0$, and therefore any pair $r_1 \in \mathbb{N}$, $r_2 \in \mathbb{N}_0$ gives a non-zero solution. Thus $\mathcal{W}_2 = \{(\varnothing, 21), (\{1\}, 21)\}$.

For $\sigma \in S_{2d'}$, let $\mathrm{Asc}(\sigma) := \{i \in [2d'] \mid \sigma(i) < \sigma(i+1)\}$ and $\mathrm{Des}(\sigma) := \{i \in [2d'] \mid \sigma(i) > \sigma(i+1)\}$. Let $\boldsymbol{r}$ and $\boldsymbol{s}$ be short for $r_1, \ldots, r_d$ and $s_1, \ldots, s_{d'}$ respectively.

Definition 3.13. Write $\mathbb{N}_0^{d+d'} = \{(\boldsymbol{r},\boldsymbol{s}) \mid r_i, s_j \in \mathbb{N}_0\}$. For $(I,\sigma) \in \mathcal{W}_d$, the set $G_{I,\sigma}$ is the set of tuples $(\boldsymbol{r},\boldsymbol{s}) \in \mathbb{N}_0^{d+d'}$ that satisfy the following equations and inequalities:

$$\left\{ \begin{array}{lll} r_i > 0 & \text{for } i \in I, & (3.3) \\ r_i = 0 & \text{for } i \in [d-1]\backslash I, & (3.4) \\ \sum_{j=1}^d w^\sigma_{i,j} r_j + \sum_{j=1}^{d'} w^\sigma_{i,d+j} s_j > 0 & \text{for } i \in \mathrm{Asc}(\sigma), & (3.5) \\ \sum_{j=1}^d w^\sigma_{i,j} r_j + \sum_{j=1}^{d'} w^\sigma_{i,d+j} s_j \geqslant 0 & \text{for } i \in \mathrm{Des}(\sigma). & (3.6) \end{array} \right.$$

Example 3.14. Let $d = 2$ and $(I,\sigma) = (\{1\}, 21)$. Then $G_{I,\sigma}$ is the set of tuples $(r_1, r_2, s_1) \in \mathbb{N}_0^3$ such that $r_1 > 0$ and $r_1 + 2r_2 - s_1 \geqslant 0$.

Remark 3.15. The set $\mathcal{W}_d$ is designed in order for the sets $G_{I,\sigma}$ to be non-empty.

Remark 3.16. The dimension of $G_{I,\sigma}$ is $d + d' - |[d-1]\backslash I|$. Therefore

$$\max\{\dim G_{I,\sigma} \mid (I,\sigma) \in \mathcal{W}_d\} = d + d',$$

and the maximum is attained for the pairs $(I,\sigma) \in \mathcal{W}_d$ with $I = [d-1]$.

We describe which entries s_j of $(\boldsymbol{r}, \boldsymbol{s})$ are positive when $(\boldsymbol{r}, \boldsymbol{s}) \in G_{I,\sigma}$.

Definition 3.17. For $\sigma \in \mathcal{S}_{2d'}$, define

$$J_\sigma := \{j \in [d'-1] \mid \sigma^{-1}(j) < \sigma^{-1}(j+1)\}.$$

Example 3.18. Let $d = 2$ and $(I,\sigma) = (\{1\}, 21)$. Then $J_\sigma = \varnothing$.

Proposition 3.19. *Let $(\boldsymbol{r}, \boldsymbol{s}) \in G_{I,\sigma}$. If $j \in J_\sigma$, then $s_j > 0$. If $j \in [d'-1]\backslash J_\sigma$, then $s_j = 0$.*

Proof. Let $j \in J_\sigma$, i.e. $\sigma^{-1}(j) < \sigma^{-1}(j+1)$. Then summing the common left-hand side of (3.5) and (3.6) over $i \in [\sigma^{-1}(j), \sigma^{-1}(j+1)-1]$ results in s_j. There necessarily is an ascent in the interval $[\sigma^{-1}(j), \sigma^{-1}(j+1)-1]$. Therefore summing (3.5) over $i \in [\sigma^{-1}(j), \sigma^{-1}(j+1)-1] \cap \mathrm{Asc}(\sigma)$ and (3.6) over $i \in [\sigma^{-1}(j), \sigma^{-1}(j+1)-1] \cap \mathrm{Des}(\sigma)$ results in $s_j > 0$. Now let $j \in [d'-1]\backslash J_\sigma$, i.e. $\sigma^{-1}(j) > \sigma^{-1}(j+1)$. Then summing the common left-hand side of (3.5) and (3.6) over $i \in [\sigma^{-1}(j+1), \sigma^{-1}(j)-1]$ results in $-s_j$. Because $\sigma \in \mathcal{S}_{2d'}$, there can only be descents the interval $[\sigma^{-1}(j+1), \sigma^{-1}(j)-1]$. Thus summing (3.6) over $i \in [\sigma^{-1}(j+1), \sigma^{-1}(j)-1]$ results in $-s_j \geqslant 0$, from which we deduce $s_j = 0$. $\square$

3.3. **Alternative description of $G_{I,\sigma}$.** In (2.7) we defined subsets $I_{E,A,C}$ of E, where A and C encoded which entries were positive and non-negative, respectively. We now describe $G_{I,\sigma}$ using such a set $I_{E,A,C}$ where $E = E_\sigma$ from Section 3.1.

Definition 3.20. For $\sigma \in \mathcal{S}_{2d'}$, let $\{j_i \mid i \in [r_\sigma]\} = R_\sigma$ with $j_1 \geqslant \cdots \geqslant j_{r_\sigma}$. For every $(I,\sigma) \in \mathcal{W}_d$, let $A_{I,\sigma}$ and $C_{I,\sigma}$ be the following subsets of $[m]$:

$$\begin{aligned} A_{I,\sigma} &:= I \cup (d + J_\sigma) \cup \left(d + d' + \{i \in [r_\sigma] \mid j_i \in \mathrm{Asc}(\sigma)\}\right), \\ C_{I,\sigma} &:= I \cup (d + J_\sigma) \cup \left\{d, d+d'\right\} \cup \left(d + d' + [r_\sigma]\right). \end{aligned}$$

Example 3.21. Let $d = 2$ and $(I,\sigma) = (\{1\}, 21)$. Then $A_{\{1\},21} = \{1\}$ and $C_{\{1\},21} = \{1,2,3,4\}$.

Proposition 3.22. *Let $(I,\sigma) \in \mathcal{W}_d$. Let $\mathrm{pr} : \mathbb{N}^{m_\sigma} \to \mathbb{N}^{d+d'}$ be the projection map which ignores the last r_σ coordinates. Restricting this projection map to the subset $I_{E_\sigma,A_{I,\sigma},C_{I,\sigma}} \subseteq E_\sigma \subseteq \mathbb{N}^{m_\sigma}$ results in a bijection between $I_{E_\sigma,A_{I,\sigma},C_{I,\sigma}}$ and $G_{I,\sigma}$.*

Proof. Let $\boldsymbol{\gamma}$ be short for $\gamma_1, \ldots, \gamma_{r_\sigma}$. Suppose that $(\boldsymbol{r}, \boldsymbol{s}, \boldsymbol{\gamma}) \in I_{E_\sigma,A_{I,\sigma},C_{I,\sigma}}$. Then (3.3) is satisfied because $I \subseteq A_{I,\sigma}$ and (3.4) is satisfied because $([d-1]\backslash I) \cap C_{I,\sigma} = \varnothing$. Also (3.6) is satisfied for all $i \in R_\sigma$ because of the definition of Φ_σ and $d + d' + [r_\sigma] \subseteq C_{I,\sigma}$. If i is, moreover, an ascend, then (3.5) holds because $d + d' + \{i \in [r_\sigma] \mid j_i \in \mathrm{Asc}(\sigma)\} \in A_{I,\sigma}$. If $i \in \mathrm{Des}(\sigma)\backslash R_\sigma$, i.e. $\sigma(i+1) < \sigma(i) \in [d']$, then $\sigma(i+1) + 1 = \sigma(i)$ (because $\sigma \in \mathcal{S}_{2d'}$) and therefore (3.6) simplifies to $-s_{\sigma(i+1)} \geqslant 0$. As $\sigma(i+1)$ is in $[d'-1]\backslash J_\sigma$ and therefore $d+\sigma(i+1)$ is not in $C_{I,\sigma}$, it follows that $s_{\sigma(i+1)} = 0$ and therefore (3.6) holds. If $i \in \mathrm{Asc}(\sigma)\backslash R_\sigma$, i.e. $\sigma(i) < \sigma(i+1) \leqslant d'$, then (3.5) simplifies to $\sum_{j=\sigma(i)+1}^{\sigma(i+1)} s_j > 0$. If $\sigma^{-1}(\sigma(i)+1) < i$, then $\sigma(i)+1 \in J_\sigma$ and therefore $d + \sigma(i) + 1 \in d + J_\sigma \subseteq A_{I,\sigma}$ and $s_{\sigma(i)+1} > 0$. If $\sigma^{-1}(\sigma(i)+1) > i$, then $\sigma(i) \in J_\sigma$ and therefore $d + \sigma(i) \in d + J_\sigma \subseteq A_{I,\sigma}$ and $s_{\sigma(i)} > 0$. In any case (3.5) holds. Thus we have that $(\boldsymbol{r}, \boldsymbol{s}) \in G_{I,\sigma}$.

The restricted projection map $\mathrm{pr}|_{I_{E_\sigma,A_{I,\sigma},C_{I,\sigma}}}$ is injective because $\boldsymbol{\gamma}$ are slack variables, therefore are uniquely determined by $(\boldsymbol{r},\boldsymbol{s})$. To prove that it is surjective, let $(\boldsymbol{r},\boldsymbol{s}) \in G_{I,\sigma}$. Again because $\boldsymbol{\gamma}$ are slack variables, we can find $(\gamma_j)_{j\in[r_\sigma]}$ such that $(\boldsymbol{r},\boldsymbol{s},\boldsymbol{\gamma}) \in E_\sigma$. Because of (3.3) and (3.4), we know that for $i \in [d-1]$, $r_i > 0$ if and only if $i \in A_{I,\sigma}$ and otherwise $i \notin C_{I,\sigma}$. Using Proposition 3.19, we find that for $j \in [d'-1]$, $s_j > 0$ if and only if $d+j \in (d+J_\sigma) \subseteq A_{I,\sigma}$ and otherwise $d+i \notin C_{I,\sigma}$. We conclude that $(\boldsymbol{r},\boldsymbol{s},\boldsymbol{\gamma}) \in I_{E_\sigma,A_{I,\sigma},C_{I,\sigma}}$. □

Let $\mathbf{X} = (X_i)_{i\in[d]}$, $\mathbf{Y} = (Y_j)_{j\in[d']}$ and $\mathbf{Z} = (Z_k)_{k\in[r_\sigma]}$ be tuples of indeterminates. The generating series enumerating the elements of E_σ, $G_{I,\sigma}$, and $I_{E_\sigma,A_{I,\sigma},C_{I,\sigma}}$ as in Section 2.5 are denoted by $E_\sigma(\mathbf{X},\mathbf{Y},\mathbf{Z})$, $G_{I,\sigma}(\mathbf{X},\mathbf{Y})$, and $I_{E_\sigma,A_{I,\sigma},C_{I,\sigma}}(\mathbf{X},\mathbf{Y},\mathbf{Z})$ respectively. By Proposition 3.22, it follows that $I_{E_\sigma,A_{I,\sigma},C_{I,\sigma}}(\mathbf{X},\mathbf{Y},\mathbf{1}) = G_{I,\sigma}(\mathbf{X},\mathbf{Y})$, where $\mathbf{1}$ is the all-one tuple of length r_σ.

Often, we will use the following subdivision of $I_{E_\sigma,A_{I,\sigma},C_{I,\sigma}}$ into simplicial monoids, which exists because of Proposition 2.29.

Definition 3.23. For every $(I,\sigma) \in \mathcal{W}_d$, let $\Gamma_{I,\sigma} = (K_u)_{u\in U_{I,\sigma}}$ be a family of simplicial monoids $K_u \subseteq \mathbb{N}_0^{m_\sigma}$ such that

- $I_{E_\sigma,A_{I,\sigma},C_{I,\sigma}} = \bigcup_{u\in U_{I,\sigma}} \overline{K}_u$ and this union is disjoint,
- $\mathrm{CF}(K_u) \subseteq \mathrm{CF}(E_\sigma)$ for all $u \in U$.

Example 3.24. Let $(I,\sigma) = (\varnothing, 21)$. Then $m_\sigma = 4$ and E_σ contains all tuples $(r_1,r_2,s_1,\gamma_1) \in \mathbb{N}_0^4$ such that $r_1 + 2r_2 - s_1 - \gamma_1 = 0$. Moreover, $A_{I,\sigma} = \varnothing$ and $C_{I,\sigma} = \{2,3,4\}$. It follows that $I_{E_\sigma,A_{I,\sigma},C_{I,\sigma}}$ contains all tuples $(r_1,r_2,s_1,\gamma_1) \in \mathbb{N}_0^4$ such that $2r_2 - s_1 - \gamma_1 = 0$ and $r_1 = 0$. One possible $\Gamma_{I,\sigma} = \{K_u \mid u \in U_{I,\sigma}\}$ is the following family (K_0,K_1,K_2,K_3):

$$\begin{aligned}
K_0 &= \{(0,0,0,0)\},\\
K_1 &= \{(0,r_2,s_1,0) \in \mathbb{N}_0^4 \mid 2r_2 - s_1 = 0\},\\
K_2 &= \{(0,r_2,0,\gamma_1) \in \mathbb{N}_0^4 \mid 2r_2 - \gamma_1 = 0\},\\
K_3 &= \{(0,r_2,s_1,\gamma_1) \in \mathbb{N}_0^4 \mid 2r_2 - s_1 - \gamma_1 = 0\},
\end{aligned}$$

where K_3 is simplicial because it has quasigenerators $(0,1,2,0)$ and $(0,1,0,2)$.

3.4. The monoid $E_{\mathrm{n.o.}} \subseteq \mathbb{N}_0^{d+d'+1}$.

We define a monoid $E_{\mathrm{n.o.}} \subseteq \mathbb{N}_0^{d+d'+1}$, again via a matrix $\Phi_{\mathrm{n.o.}}$. We list its completely fundamental elements, define a specific subset E_0 and describe a specific triangulation of the cone $\mathcal{C}_{E_{\mathrm{n.o.}}}$ generated by $E_{\mathrm{n.o.}}$.

Definition 3.25. Let $\Phi_{\mathrm{n.o.}}$ be the $1 \times (d+d'+1)$ matrix

$$\Phi_{\mathrm{n.o.}} := [0^{(d-2)}, 1, 2, (-1)^{(d'+1)}]. \tag{3.7}$$

Let $E_{\mathrm{n.o.}} \subseteq \mathbb{N}_0^{d+d'+1}$ be the monoid $E_{\Phi_{\mathrm{n.o.}}}$ as in (2.4).

Recall from Remark 2.15 that the convex cone $\mathcal{C}_{E_{\mathrm{n.o.}}}$ generated by $E_{\mathrm{n.o.}}$ is a pointed convex polyhedral cone. The completely fundamental elements of $E_{\mathrm{n.o.}}$ each lie on an extreme ray of $\mathcal{C}_{E_{\mathrm{n.o.}}}$. Therefore by Theorem 2.18, the completely fundamental elements of $E_{\mathrm{n.o.}}$ correspond to the minimal (non-empty) supports in $L(E_{\mathrm{n.o.}})$. By (3.7), we see that the minimal (non-empty) supports in $L(E_{\mathrm{n.o.}})$ are

$$\begin{cases}
\{i\} & \text{for } i \in [d-2]; & (3.8)\\
\{d-1, i\} & \text{for } i \in d + [d'+1]; & (3.9)\\
\{d, i\} & \text{for } i \in d + [d'+1]. & (3.10)
\end{cases}$$

For $i \in [d+d'+1]$, let $\delta_i \in \mathbb{N}_0^{d+d'+1}$ be the ith unit basis vector. The $2d'+d$ completely fundamental elements of $E_{\text{n.o.}}$ are the following:

$$\begin{cases} \delta_i & \text{for } i \in [d-2]; & (3.11)\\ \delta_{d-1} + \delta_i & \text{for } i \in d + [d'+1]; & (3.12)\\ \delta_d + 2\delta_i & \text{for } i \in d + [d'+1]. & (3.13) \end{cases}$$

Special attention will go to one specific completely fundamental element, namely $\delta_d + 2\delta_{d+d'+1}$.

By Theorem 2.18, the 2-faces of $\mathcal{C}_{E_{\text{n.o.}}}$ can be found by looking at the elements of $L(E_{\text{n.o.}})$ that contain at least one of the sets in (3.8)-(3.10), yet do not strictly contain any non-empty elements of $L(E_{\text{n.o.}})$ that are not listed in (3.8)-(3.10). We are especially interested in the 2-faces of $\mathcal{C}_{E_{\text{n.o.}}}$ that contain $\delta_d + 2\delta_{d+d'+1}$. The elements of $L(E_{\text{n.o.}})$ that contain $\{d, d+d'+1\}$ and do not strictly contain any non-empty elements of $L(E_{\text{n.o.}})$ not listed in (3.8)-(3.10) are the following:

$$\begin{cases} \{i, d, d+d'+1\} & \text{for } i \in [d-2]; & (3.14)\\ \{d-1, d, d+d'+1\}; & & (3.15)\\ \{d, i, d+d'+1\} & \text{for } i \in d + [d']. & (3.16) \end{cases}$$

Note that the sets $\{d-1, d, i, d+d'+1\}$ for $i \in d + [d']$ are elements of $L(E_{\text{n.o.}})$, but they strictly contain the set $\{d-1, d, d+d'+1\}$, which is not listed in (3.8)-(3.10), and therefore they do not correspond to a 2-face. The 2-faces of $\mathcal{C}_{E_{\text{n.o.}}}$ that contain $\delta_d + 2\delta_{d+d'+1}$ are thus generated by

$$\begin{cases} \{\delta_d + 2\delta_{d+d'+1}, \delta_i\} & \text{for } i \in [d-2]; & (3.17)\\ \{\delta_d + 2\delta_{d+d'+1}, \delta_{d-1} + \delta_{d+d'+1}\}; & & (3.18)\\ \{\delta_d + 2\delta_{d+d'+1}, \delta_d + 2\delta_i\} & \text{for } i \in d + [d']. & (3.19) \end{cases}$$

Definition 3.26. Let $\mathcal{C}_0$ be the subcone (not a face) of $\mathcal{C}_{E_{\text{n.o.}}}$ generated by the set

$$\{\delta_i \mid i \in [d-2]\} \cup \{\delta_{d-1} + \delta_{d+d'+1}\} \cup \{\delta_d + 2\delta_i \mid i \in d + [d'+1]\},$$

and let $E_0 = \mathcal{C}_0 \cap E_{\text{n.o.}}$.

Remark 3.27. Being generated by $d + d'$ linearly independent elements of $E_{\text{n.o.}}$, $\mathcal{C}_0$ and E_0 are simplicial and have dimension $d + d'$.

Proposition 3.28 (adaptation of [14, Lem. 4.5.1])**.** *The pointed polyhedral cone $\mathcal{C}_{E_{\text{n.o.}}}$ has a triangulation $\Gamma = (\mathcal{K}_u)_{u \in U}$ whose one-dimensional faces are the extreme rays of $\mathcal{C}_{E_{\text{n.o.}}}$ and there is a $u \in U$ with $\mathcal{K}_u = \mathcal{C}_0$.*

Proof. Use the algorithm in the proof of [14, Lem. 4.5.1], while ordering the extreme rays generated by each of the elements of (3.26) that are not $\delta_d + 2\delta_{d+d'+1}$ first. □

Remark 3.29. By definition, $\mathcal{C}_0$ contains all 2-faces of $\mathcal{C}_{E_{\text{n.o.}}}$ that contain $\delta_d + 2\delta_{d+d'+1}$. Therefore in the triangulation Γ from Proposition 3.28, the faces of $\mathcal{C}_0$ are the only elements that contain $\delta_d + 2\delta_{d+d'+1}$.

3.5. **The subsets $H_{I,J} \subseteq \mathbb{N}_0^{d+d'}$.** We define subsets $H_{I,J} \subseteq \mathbb{N}_0^{d+d'}$ for every $I \subseteq [d-1]$ and $J \subseteq [d'-1]$. These subsets are used in Section 5.2 to write down a formula for $\zeta^{\text{n.o.}}_{\mathfrak{f}_{2,d}(\mathfrak{o})}(s)$.

Definition 3.30. For every $I \subseteq [d-1]$ and $J \subseteq [d'-1]$, let $H_{I,J}$ be the set of tuples $(\boldsymbol{r}, \boldsymbol{s}) \in \mathbb{N}_0^{d+d'}$ that satisfy the following equations and inequalities:

$$\begin{cases} r_i > 0 & \text{for } i \in I, & (3.20)\\ r_i = 0 & \text{for } i \in [d-1]\setminus I, & (3.21)\\ s_j > 0 & \text{for } j \in J, & (3.22)\\ s_j = 0 & \text{for } j \in [d'-1]\setminus J, & (3.23)\\ r_{d-1} + 2r_d - \sum_{j=1}^{d'} s_j \geqslant 0. & & (3.24)\end{cases}$$

Example 3.31. Let $d = 2$, $I = \{1\}$, and $J = \varnothing$. Then $H_{I,J}$ is the set of tuples $(r_1, r_2, s_1) \in \mathbb{N}_0^3$ such that $r_1 > 0$ and $r_1 + 2r_2 - s_1 \geqslant 0$.

Just as Proposition 3.22 described $G_{I,\sigma}$ as a set of the form $I_{E_\sigma,A,C}$, we now describe the sets $H_{I,J}$ as sets of the form $I_{E_{\text{n.o.}},A,C}$.

Definition 3.32. For every $I \subseteq [d-1]$ and $J \subseteq [d'-1]$, let $A_{I,J}$ and $C_{I,J}$ be the following subsets of $[d + d' + 1]$:

$$A_{I,J} := I \cup (d + J),$$
$$C_{I,J} := I \cup (d + J) \cup \{d, d + d', d + d' + 1\}.$$

Example 3.33. Let $d = 2$, $I = \{1\}$, and $J = \varnothing$. Then $A_{\{1\},\varnothing} = \{1\} \cup \varnothing$ and $C_{\{1\},\varnothing} = \{1\} \cup \varnothing \cup \{2, 3, 4\}$.

Proposition 3.34. *Let* $\mathrm{pr} : \mathbb{N}^{d+d'+1} \to \mathbb{N}^{d+d'}$ *be the projection map which ignores the last coordinate. For every* $I \subseteq [d-1]$ *and* $J \subseteq [d'-1]$*, restricting this projection map to the subset* $I_{E_{\text{n.o.}},A_{I,J},C_{I,J}} \subseteq E_{\text{n.o.}} \subseteq \mathbb{N}^{d+d'+1}$ *results in a bijection between* $I_{E_{\text{n.o.}},A_{I,J},C_{I,J}}$ *and* $H_{I,J}$*.*

Proof. Suppose that $(\boldsymbol{r}, \boldsymbol{s}, \gamma_1) \in I_{E_{\text{n.o.}},A_{I,J},C_{I,J}}$. Then (3.20) is satisfied because $I \subseteq A_{I,J}$ and (3.21) is satisfied because $([d-1]\setminus I) \cap C_{I,J} = \varnothing$. Also (3.22) is satisfied because $(d + J) \subseteq A_{I,J}$ and (3.23) is satisfied because $(d + ([d'-1]\setminus J)) \cap C_{I,J} = \varnothing$. Lastly (3.24) follows from the definition of $\Phi_{\text{n.o.}}$. Thus $(\boldsymbol{r}, \boldsymbol{s}) \in H_{I,J}$.

Suppose that $(\boldsymbol{r}, \boldsymbol{s}) \in H_{I,J}$. Let $\gamma_1 = r_{d-1} + 2r_d - \sum_{j=1}^{d'} s_j$. Then $(\boldsymbol{r}, \boldsymbol{s}, \gamma_1) \in I_{E_{\text{n.o.}},A_{I,J},C_{I,J}}$. Moreover, this is the unique element $\gamma_1 \in \mathbb{N}_0$ such that $(\boldsymbol{r}, \boldsymbol{s}, \gamma_1) \in I_{E_{\text{n.o.}},A_{I,J},C_{I,J}} \subseteq E_{\text{n.o.}}$, because if $(\boldsymbol{r}, \boldsymbol{s}, \gamma_1) \in E_{\text{n.o.}}$, then $r_{d-1} + 2r_d - \sum_{j=1}^{d'} s_j - \gamma_1 = 0$. □

Remark 3.35. Note that

$$E_{\text{n.o.}} = \bigcup_{I \subseteq [d-1], J \subseteq [d'-1]} I_{E_{\text{n.o.}},A_{I,J},C_{I,J}},$$

and this union is disjoint.

We record the following reciprocity result for the generating functions $H_{I,J}(\mathbf{X}, \mathbf{Y})$.

Proposition 3.36. *Let* $K \subseteq [d-1]$ *and* $L \subseteq [d'-1]$*, then*

$$\sum_{\substack{I \subseteq [d-1], I \supseteq K,\\ J \subseteq [d'-1], J \supseteq L}} H_{I,J}(\mathbf{X}^{-1}, \mathbf{Y}^{-1}) = (-1)^{d+d'} \sum_{\substack{I \subseteq [d-1], I \supseteq [d-1]\setminus K,\\ J \subseteq [d'-1], J \supseteq [d'-1]\setminus L}} X_d Y_{d'} H_{I,J}(\mathbf{X}, \mathbf{Y}).$$

Proof. By Proposition 3.34 and (2.12),

$$H_{I,J}(\mathbf{X}^{-1}, \mathbf{Y}^{-1}) = I_{E_{\text{n.o.}},A_{I,J},C_{I,J}}(\mathbf{X}^{-1}, \mathbf{Y}^{-1}, 1) = \sum_{\substack{B \in L(E_{\text{n.o.}}),\\ A_{I,J} \subseteq B \subseteq C_{I,J}}} \overline{F}_{E_{\text{n.o.}},B}(\mathbf{X}^{-1}, \mathbf{Y}^{-1}, 1).$$

Therefore summing over $I \supseteq K$ and $J \supseteq L$ results in

$$\sum_{I \supseteq K, J \supseteq L} H_{I,J}(\mathbf{X}^{-1}, \mathbf{Y}^{-1}) = \sum_{B \in L(E_{\text{n.o.}}), A_{K,L} \subseteq B} \overline{F}_{E_{\text{n.o.}},B}(\mathbf{X}^{-1}, \mathbf{Y}^{-1}, 1).$$

Note that any subset of $[d + d' + 1]$ that contains d and $d + d'$ is an element of the lattice of supports $L(E_{\text{n.o.}})$. In particular, $[d + d' + 1] \backslash A_{K,L}$ is an element of $L(E_{\text{n.o.}})$. Therefore, we may use Corollary 2.27 to obtain

$$\sum_{I \supseteq K, J \supseteq L} H_{I,J}(\mathbf{X}^{-1}, \mathbf{Y}^{-1}) = (-1)^{\dim(E_{\text{n.o.}})} \sum_{\substack{D \in L(E_{\text{n.o.}}), \\ D \supseteq [d+d'+1] \backslash A_{K,L}}} \overline{F}_{E_{\text{n.o.}},D}(\mathbf{X}, \mathbf{Y}, 1).$$

The dimension of $E_{\text{n.o.}}$ is $d + d'$ because it is a subset of $\mathbb{N}^{d+d'+1}$ subjected to one linear equation. If $D \in L(E_{\text{n.o.}})$ with $D \supseteq [d + d' + 1] \backslash A_{K,L}$, then there are unique subsets $I \subseteq [d - 1]$ and $J \subseteq [d - 1]$ containing $[d - 1] \backslash K$ and $[d' - 1] \backslash L$ respectively such that $D = C_{I,J}$ and vice versa. Thus

$$\sum_{I \supseteq K, J \supseteq L} H_{I,J}(\mathbf{X}^{-1}, \mathbf{Y}^{-1}) = (-1)^{d+d'} \sum_{\substack{I \supseteq [d-1] \backslash K, \\ J \supseteq [d'-1] \backslash L}} \overline{F}_{E_{\text{n.o.}},C_{I,J}}(\mathbf{X}, \mathbf{Y}, 1).$$

It is easy to verify that for $I \subseteq [d - 1]$ and $J \subseteq [d' - 1]$, the map

$$\overline{F}_{E_{\text{n.o.}},C_{I,J}} \longrightarrow H_{I,J} : (a_1, \ldots, a_d, b_1, \ldots, b_{d'}, z) \longmapsto (a_1, \ldots, a_d - 1, b_1, \ldots, b_{d'} - 1)$$

is a (well-defined) bijection. Thus

$$\overline{F}_{E_{\text{n.o.}},C_{I,J}}(\mathbf{X}, \mathbf{Y}, 1) = X_d Y_{d'} H_{I,J}(\mathbf{X}, \mathbf{Y}).$$ □

3.6. **Dyck words and the relation between $H_{I,J}$ and $G_{I,\sigma}$.** We associate a Dyck word w_σ to permutations $\sigma \in \mathcal{S}_{2d'}$. Afterwards, we describe how the sets $G_{I,\sigma}$ and $H_{I,J}$ are related.

A *Dyck word of length* $2d'$ is a word w in the letters $\mathsf{0}$ and $\mathsf{1}$ such that $\mathsf{0}$ and $\mathsf{1}$ each occur d' times in w and no initial segment of w contains more ones than zeroes. For example, 001011 is a Dyck word of length 6, whereas 011001 is not as the initial segment 011 contains more ones than zeroes. We write $\mathcal{D}_{2d'}$ for the set of Dyck words of length $2d'$. The Dyck word $\mathsf{0}^{d'}\mathsf{1}^{d'} \in \mathcal{D}_{2d'}$ is called the *trivial Dyck word of length* $2d'$.

Definition 3.37. Let $\sigma \in \mathcal{S}_{2d'}$. The *Dyck word* w_σ *associated with* σ is the Dyck word of length $2d'$ where for each $i \in [2d']$, the i-th letter of w_σ is $\mathsf{0}$ if $\sigma(i) > d'$ and $\mathsf{1}$ if $\sigma(i) \leqslant d'$.

Example 3.38. Let $d = 3$ and $\sigma = 451623 \in \mathcal{S}_6$. Then $w_\sigma = \mathsf{001011} = \mathsf{0}^2\mathsf{101}^2$.

Remark 3.39. Note that w_σ is indeed a Dyck word because $\sigma \in \mathcal{S}_{2d'}$. It would not necessarily be a Dyck word if σ was a general permutation in $S_{2d'}$.

The following proposition links the sets $G_{I,\sigma}$ and $H_{I,J}$. Recall the definition of J_σ in Definition 3.17.

Proposition 3.40. *Let $I \subseteq [d - 1]$, $J \subseteq [d' - 1]$, and*

$$\mathcal{S}_{I,J} := \{\sigma \in \mathcal{S}_{2d'} \mid (I, \sigma) \in \mathcal{W}_d, J_\sigma = J, w_\sigma = \mathsf{0}^{d'}\mathsf{1}^{d'}\}.$$

Then $H_{I,J} = \bigcup_{\sigma \in \mathcal{S}_{I,J}} G_{I,\sigma}$ and this union is disjoint.

Proof. Suppose that $\sigma \in \mathcal{S}_{I,J}$ and $(\boldsymbol{r}, \boldsymbol{s}) \in G_{I,\sigma}$. Then (3.20) and (3.21) hold because of (3.3) and (3.4). Using Proposition 3.19, we find that (3.22) and (3.23) hold. Lastly, (3.24) holds because (3.24) is what (3.6) reduces to when $w_\sigma = \mathsf{0}^{d'}\mathsf{1}^{d'}$ and $i = d'$. Thus $(\boldsymbol{r}, \boldsymbol{s}) \in H_{I,J}$.

Conversely, suppose that $(\boldsymbol{r}, \boldsymbol{s}) \in H_{I,J}$. Let $\sigma \in S_{2d'}$ be the unique permutation such that

$$\sum_{j=1}^{d} v_{\sigma(i),j} r_j + \sum_{j=1}^{d'} v_{\sigma(i),d+j} s_j \geqslant \sum_{j=1}^{d} v_{\sigma(i+1),j} r_j + \sum_{j=1}^{d'} v_{\sigma(i+1),d+j} s_j$$

for every $i \in [2d'-1]$ and $\sigma(i) > \sigma(i+1)$ if equality holds. The inequality (3.24) implies that $\{\sigma(i) \mid i \in [d']\} = d' + [d']$ and $\{\sigma(i) \mid i \in d' + [d']\} = [d']$, therefore $w_\sigma = 0^{d'} 1^{d'}$. It also follows by this fact and the construction of σ that $\sigma \in \mathcal{S}_{2d'}$. Also $(I, \sigma) \in \mathcal{W}_{2d'}$ because if $\boldsymbol{r}$ is non-zero, then it is a non-zero solution to (3.2), and if $\boldsymbol{r}$ is zero, then $I = \varnothing$, $\sigma(i) = 2d' + 1 - i$ for $i \in [d']$, and $\delta_{d'}$ is a non-zero solution to (3.2). If $j \in J_\sigma$, then summing the right-hand side minus the left-hand side of (3.6) over all $i \in [\sigma^{-1}(j), \sigma^{-1}(j+1) - 1]$ results in $s_j > 0$. Therefore $J_\sigma \subseteq J$. Similarly if $j \in [d'-1] \backslash J_\sigma$, then summing the right-hand side minus the left-hand side of (3.6) over all $i \in [\sigma^{-1}(j+1), \sigma^{-1}(j) - 1]$ results in $-s_j \geqslant 0$. Therefore $[d'-1] \backslash J_\sigma \subseteq [d'-1] \backslash J$ and we conclude that $J_\sigma = J$. Obviously (3.3) and (3.4) hold because of (3.20) and (3.21). Moreover, (3.5) and (3.6) hold by construction of σ. Thus $(\boldsymbol{r}, \boldsymbol{s}) \in G_{I,\sigma}$.

The disjointness follows from the definition of $G_{I,\sigma}$. □

4. The subalgebra zeta function of $\mathfrak{f}_{2,d}$

The main result in this section is Theorem 4.24, which gives an explicit formula for $\zeta_{\mathfrak{f}_{2,d}(\mathfrak{o})}(s)$. The remainder of the section works towards proving this formula.

4.1. An explicit infinite sum formula for $\zeta_{\mathfrak{f}_{2,d}(\mathfrak{o})}(s)$.

First, we derive Proposition 4.3, which provides an explicit infinite sum formula for $\zeta_{\mathfrak{f}_{2,d}(\mathfrak{o})}(s)$. It consists of an infinite summation over a subset of $\mathbb{N}_0^{d+d'}$ with $d' := \binom{d}{2}$, where each summand is a product of Gaussian binomial coefficients and a power of $q_\mathfrak{o}$. Recall that $\mathcal{P}_n$ is the set of integer partitions of at most n (non-zero) parts, i.e. the set of tuples $\lambda = (\lambda_i)_{i \in [n]} \in \mathbb{N}_0^n$ with $\lambda_i \geqslant \lambda_{i+1}$ for $i \in [n-1]$. We start by associating a partition μ_λ of length d' with every partition λ of length d.

Definition 4.1. Given $\lambda \in \mathcal{P}_d$, let $\mu_\lambda \in \mathcal{P}_{d'}$ be the integer partition $(\mu_j)_{j \in [d']}$ such that the multisets $\{\mu_j \mid j \in [d']\}$ and $\{\lambda_i + \lambda_{i'} \mid i < i' \in [d]\}$ coincide.

Informally speaking, the integers $\mu_1, \dots, \mu_{d'}$ are the integers $\lambda_i + \lambda_{i'}$ brought into non-ascending order.

Example 4.2. If $\lambda = (3,2,2,0)$, then $\mu_\lambda = (5,5,4,3,2,2)$.

Recall that $\lambda_1^{(n)} := (\lambda_1)_{i \in [n]} \in \mathcal{P}_n$ and $|(\lambda_i)_{i \in [n]}| := \sum_{i=1}^n \lambda_i$.

Proposition 4.3. *For all $d \in \mathbb{N}_{\geqslant 2}$ and cDVR $\mathfrak{o}$,*

$$\zeta_{\mathfrak{f}_{2,d}(\mathfrak{o})}(s) = \sum_{\lambda \in \mathcal{P}_d} \sum_{\nu \in \mathcal{P}_{d'}, \nu \leqslant \mu_\lambda} \alpha\left(\lambda_1^{(n)}, \lambda; \mathfrak{o}\right) \alpha(\mu_\lambda, \nu; \mathfrak{o}) q_\mathfrak{o}^{-s|\lambda|} q_\mathfrak{o}^{(d-s)|\nu|}, \tag{4.1}$$

where $\alpha(\lambda, \mu; \mathfrak{o})$ is discussed in Section 2.2.1.

Proof. Recall that, as an $\mathfrak{o}$-module, $\mathfrak{f}_{2,d}(\mathfrak{o})$ is generated by $\{x_i \mid i \in [d]\} \cup \{[x_i, x_j] \mid i, j \in [d]\}$. Let $\mathcal{L}_1$ be the rank-d submodule generated by $\{x_i \mid i \in [d]\}$ and $\mathcal{L}_2$ be the rank-d' submodule generated by $\{[x_i, x_j] \mid i, j \in [d]\}$. Hence $\mathfrak{f}_{2,d} = \mathcal{L}_1 \oplus \mathcal{L}_2$.

Given a submodule $\Lambda \leqslant \mathfrak{f}_{2,d}(\mathfrak{o})$, we associate two submodules Λ_1 and Λ_2 of $\mathcal{L}_1$ and $\mathcal{L}_2$ respectively. The second submodule, Λ_2, is $\Lambda \cap \mathcal{L}_2$, while the first submodule Λ_1 is the unique submodule $\Lambda_1 \leqslant \mathcal{L}_1$ such that $(\Lambda_1 \oplus \mathcal{L}_2)/(\mathcal{L}_1) = (\Lambda)/(\mathcal{L}_1)$. This way Λ is not necessarily equal to $\Lambda_1 \oplus \Lambda_2$, but it always holds that the index $|\mathfrak{f}_{2,d}(\mathfrak{o}) : \Lambda|$ is

$|\mathcal{L}_1 : \Lambda_1| \cdot |\mathcal{L}_2 : \Lambda_2|$. The condition that Λ is a subalgebra of $\mathfrak{f}_{2,d}(\mathfrak{o})$ is equivalent to $[\Lambda_1, \Lambda_1]$ being a submodule of Λ_2. It follows from [8, Lemma 6.1] that

$$\zeta_{\mathfrak{f}_{2,d}(\mathfrak{o})}(s) = \sum_{\Lambda_1 \leqslant \mathcal{L}_1} \sum_{\substack{\Lambda_2 \leqslant \mathcal{L}_2 \\ [\Lambda_1,\Lambda_1] \leqslant \Lambda_2}} |\mathcal{L}_1 : \Lambda_1|^{-s} |\mathcal{L}_2 : \Lambda_2|^{d-s}.$$

Recall Definition 2.8 of the elementary divisor type $\varepsilon(\Lambda)$ of a submodule Λ. We write the zeta function as a sum over the elementary divisor type of Λ_1:

$$\zeta_{\mathfrak{f}_{2,d}(\mathfrak{o})}(s) = \sum_{\lambda \in \mathcal{P}_d} \sum_{\substack{\Lambda_1 \leqslant \mathcal{L}_1 \\ \varepsilon(\Lambda_1) = \lambda}} |\mathcal{L}_1 : \Lambda_1|^{-s} \sum_{\substack{\Lambda_2 \leqslant \mathcal{L}_2 \\ [\Lambda_1,\Lambda_1] \leqslant \Lambda_2}} |\mathcal{L}_2 : \Lambda_2|^{d-s}.$$

Let λ be the elementary divisor type $\varepsilon(\Lambda_1)$ of $\Lambda_1 \leqslant \mathcal{L}_1$. Then $\{\pi^{\lambda_i + \lambda_j} \mid i < j \in [d]\}$ yields the elementary divisor type of $[\Lambda_1, \Lambda_1] \leqslant \mathcal{L}_2$. Thus by Definition 4.1, $\mu_{\varepsilon(\Lambda_1)}$ is the elementary divisor type $\varepsilon([\Lambda_1, \Lambda_1])$ of $[\Lambda_1, \Lambda_1] \leqslant \mathcal{L}_2$. When counting submodules $\Lambda_2 \leqslant \mathcal{L}_2$ that contain a given submodule $[\Lambda_1, \Lambda_1]$, the only thing that is important is the elementary divisor type of the given submodule $[\Lambda_1, \Lambda_1]$, see Proposition 2.10. Since the elementary divisor type of $[\Lambda_1, \Lambda_1]$ is completely determined by $\varepsilon(\Lambda_1)$, we can conclude that counting submodules Λ_2 that contain a given submodule $[\Lambda_1, \Lambda_1]$ is also completely determined $\varepsilon(\Lambda_1) = \lambda$. Therefore, the last two summations in (4.1) are independent counting problems, connected by the elementary divisor type of Λ_1:

$$\zeta_{\mathfrak{f}_{2,d}(\mathfrak{o})}(s) = \sum_{\lambda \in \mathcal{P}_d} \left(\sum_{\substack{\Lambda_1 \leqslant \mathcal{L}_1 \\ \varepsilon(\Lambda_1) = \lambda}} |\mathcal{L}_1 : \Lambda_1|^{-s} \right) \left(\sum_{\substack{\Lambda_2 \leqslant \mathcal{L}_2 \\ M_\lambda \leqslant \Lambda_2}} |\mathcal{L}_2 : \Lambda_2|^{d-s} \right), \tag{4.2}$$

where M_λ is any submodule of $\mathcal{L}_2$ that has elementary divisor type μ_λ.

The first counting problem (the first brackets), pertains to counting submodules $\Lambda_1 \leqslant \mathcal{L}_1$ with fixed elementary divisor type. The solution to this first counting problem is discussed in Proposition 2.9. The result in this case is

$$\sum_{\substack{\Lambda_1 \leqslant \mathcal{L}_1 \\ \varepsilon(\Lambda_1) = \lambda}} |\mathcal{L}_1 : \Lambda_1|^{-s} = \alpha(\lambda_1^{(n)}, \lambda; \mathfrak{o}) q_{\mathfrak{o}}^{-s|\lambda|}.$$

The second counting problem (the second brackets) pertains to counting submodules $\Lambda_2 \leqslant \mathcal{L}_2$ that contain a given submodule M_λ of which we know the elementary divisor type, namely μ_λ. The solution to this second counting problem is discussed in Proposition 2.10 and the result in this case is

$$\sum_{\substack{\Lambda_2 \leqslant \mathcal{L}_2 \\ M_\lambda \leqslant \Lambda_2}} |\mathcal{L}_2 : \Lambda_2|^{d-s} = \sum_{\substack{\nu \in \mathcal{P}_{d'} \\ \nu \leqslant \mu_\lambda}} \alpha(\mu_\lambda, \nu; \mathfrak{o}) q_{\mathfrak{o}}^{(d-s)|\nu|}. \qquad \square$$

The formula for $\zeta_{\mathfrak{f}_{2,d}(\mathfrak{o})}(s)$ in Proposition 4.3 is explicit, however not closed since it contains infinite sums. In the following sections, the formula will be written as a finite sum, where each summand will be a product of Gaussian multinomial coefficients and a substitution of a series of the form (2.5). This will make the formula amenable to computer algebra systems capable of enumerating integral points in polyhedra.

4.2. **The factor $\alpha(\mu_\lambda, \nu; \mathfrak{o})$ in terms of $\sigma_{\lambda,\nu}$.** We associate a permutation $\sigma_{\lambda,\nu} \in S_{2d'}$ with each pair $(\lambda, \nu) \in \mathcal{P}_d \times \mathcal{P}_{d'}$. Afterwards, we rewrite the factor $\alpha(\mu_\lambda, \nu; \mathfrak{o})$ that appeared in (4.1) as a product of a power of $q_{\mathfrak{o}}$ and a product of Gaussian binomial coefficients that depends only on $\sigma_{\lambda,\nu}$. Recall the definition of the map b in Definition 3.3.

Definition 4.4. Let $\lambda \in \mathcal{P}_d$ and $\nu \in \mathcal{P}_{d'}$. Then $\sigma_{\lambda,\nu}$ is the permutation $\sigma \in S_{2d'}$ defined inductively as follows. Consider the multiset $\Sigma = \{\lambda_i + \lambda_j\}_{i<j\in[d]} \cup \{\nu_i\}_{i\in[d']}$. Let $\sigma(1)$ be the maximal b-value among the indices of the elements of Σ which are maximal. Now assume that $i > 1$. To find $\sigma(i)$, consider the subset of Σ comprising the elements whose indices have an image under b that is not yet in $\{\sigma(j) \mid j < i\}$. Let $\sigma(i)$ be the maximal b-value among the indices of the elements of this subset which are maximal.

Example 4.5. Let $d = 3$, $\lambda = (5, 4, 1)$, and $\nu = (6, 2, 2)$. Then $\lambda_1 + \lambda_2 = 9$, $\lambda_1 + \lambda_3 = 6$, and $\lambda_2 + \lambda_3 = 5$, thus $\Sigma = \{9, 6, 5, 6, 2, 2\}$. Clearly, $\lambda_1 + \lambda_2 = 9$ is the (unique) maximal element of Σ. Therefore $\sigma(1) = b((1, 2)) = 4$. Among the remaining elements, both $\lambda_1 + \lambda_3 = 6$ and $\nu_1 = 6$ are maximal. Of the two, the index of $\lambda_1 + \lambda_3$ has a greater image under b, whence $\sigma(2) = b((1, 3)) = 5$. Among the remaining elements, $\nu_1 = 6$ is maximal, whence $\sigma(3) = b(1) = 1$. Continuing in this way, we find that $\sigma_{\lambda,\nu} = 451632$.

Recall the definition of the set $\mathcal{S}_{2d'}$ in Definition 3.1.

Remark 4.6. Let $\lambda \in \mathcal{P}_d$ and $\nu \in \mathcal{P}_{d'}$. Then $\nu \leqslant \mu_\lambda$ if and only if $\sigma_{\lambda,\nu} \in \mathcal{S}_{2d'}$.

Next, we define integers $L_j(\sigma)$ and $M_j(\sigma)$ for all $\sigma \in \mathcal{S}_{2d'}$ and $j \in \{0\} \cup [2d']$. These are related to the integers M_j and L_j from Definition 2.4 as is partially discussed in the proof of Lemma 4.9.

Definition 4.7. For $\sigma \in \mathcal{S}_{2d'}$ and $j \in \{0\} \cup [2d']$, define $L_0(\sigma) := 0$, $M_0(\sigma) := 0$,

$$\begin{aligned} L_j(\sigma) &:= \#\{\sigma(i) \mid i \in [j]\} \cap (d' + [d']) && \text{for all } j \in [2d'], \\ M_j(\sigma) &:= \#\{\sigma(i) \mid i \in [j]\} \cap [d'] && \text{for all } j \in [2d']. \end{aligned}$$

Example 4.8. Let $d = 3$, $\sigma = 451623 \in \mathcal{S}_6$, and $j = 3$. Then $L_3(\sigma) = |\{4, 5, 1\} \cap \{4, 5, 6\}| = 2$ and $M_3(\sigma) = |\{4, 5, 1\} \cap \{1, 2, 3\}| = 1$.

Recall that $\mathrm{Asc}(\sigma) := \{i \in [2d'] \mid \sigma(i) < \sigma(i + 1)\}$. The following lemma writes the product of Gaussian binomial coefficients in (2.3) in a way that only depends only on $\sigma_{\lambda,\nu}$, using the integers $L_j(\sigma)$ and $M_j(\sigma)$ from Definition 4.7.

Lemma 4.9. *Let $\lambda \in \mathcal{P}_d$ and $\nu \in \mathcal{P}_{d'}$ be integer partitions with $\nu \leqslant \mu_\lambda$. Let $\sigma := \sigma_{\lambda,\nu} \in \mathcal{S}_{2d'}$ and M_j and L_j be as in Definition 2.4 with $\mu = \mu_\lambda$. Let $r := |\mathrm{Asc}(\sigma)| + 1$ and $\{j_i \mid i \in [r - 1]\} := \mathrm{Asc}(\sigma)$ with $j_i < j_{i+1}$ for $i \in [r - 2]$. Moreover, set $j_0 := 0$ and $j_r := 2d'$. Then*

$$\prod_{j\in[2d']} \binom{L_j - M_{j-1}}{M_j - M_{j-1}}_{q_o^{-1}} = \prod_{i\in[r]} \binom{L_{j_i}(\sigma) - M_{j_{i-1}}(\sigma)}{M_{j_i}(\sigma) - M_{j_{i-1}}(\sigma)}_{q_o^{-1}}.$$

Proof. For $i \in [r]$, let k_i be the smallest element of $(j_{i-1}, j_i]$ such that $m_{k_i} = m_{k_i+1} = \cdots = m_{j_i}$. It follows directly that $L_{k_i} = L_{k_i+1} = \cdots = L_{j_i}$ and $M_{k_i} = M_{k_i+1} = \cdots = M_{j_i}$. It also follows that $m_{k_i-1} > m_{k_i}$. We claim that $M_{j_{i-1}} = M_{j_{i-1}+1} = \cdots = M_{k_i-1}$ as well. Indeed, suppose that $M_j \neq M_{j+1}$ for some $j \in [j_{i-1}, k_i - 1)$. Then $m_j > m_{j+1}$ and there is a ν_l with $\nu_l = m_{j+1}$, that is, there is a $j' \in [j + 1, k_i)$ with $\sigma(j') = l \in [d']$. By construction, $m_{j+1} \geqslant m_{k_i-1} > m_{k_i} = m_{j_i}$, thus we find that $m_j > m_{j+1} = \nu_l > m_{j_i}$. Both if $\sigma(j_i) \in [d']$ or $\sigma(j_i) \in d' + [d']$, this implies that there is an ascent of σ in the interval $[j + 1, j_i)$, which is a contradiction. Thus, we showed that the factor corresponding to $j \in [2d']$ in (4.9) is one for all $j \in (k_i, j_i]$ and $j \in (j_{i-1}, k_i)$ with $i \in [r]$. The remaining factors are

$$\binom{L_{k_i} - M_{k_i-1}}{M_{k_i} - M_{k_i-1}}_{q_o^{-1}} = \binom{L_{j_i} - M_{j_{i-1}}}{M_{j_i} - M_{j_{i-1}}}_{q_o^{-1}} \tag{4.3}$$

for all $i \in [r]$. If $j_i \in \mathrm{Asc}(\sigma)$, then $m_{j_i} > m_{j_i+1}$ and therefore $L_{j_i} = L_{j_i}(\sigma)$ and $M_{j_i} = M_{j_i}(\sigma)$. The same conclusion holds for $j_0 = 0$ and $j_r = 2d'$. Thus (4.9) holds. □

The following proposition writes the factor $\alpha(\mu_\lambda, \nu; \mathfrak{o})$ in (4.1) as a product of a power of $q_\mathfrak{o}$ and a product of Gaussian binomial coefficients that depends only on $\sigma_{\lambda,\nu}$.

Proposition 4.10. *Let $\lambda \in \mathcal{P}_d$ and $\nu \in \mathcal{P}_{d'}$ be integer partitions with $\nu \leqslant \mu_\lambda$. Let $\sigma = \sigma_{\lambda,\nu}$ and write $\{m_i \mid i \in [2d']\} := \mu_\lambda \cup \nu$ with $m_i \geqslant m_{i+1}$ for all $i \in [2d']$ as in Definition 2.4. Then*

$$\alpha(\mu_\lambda, \nu; \mathfrak{o}) = \left(\prod_{i \in [r]} \binom{L_{j_i}(\sigma) - M_{j_{i-1}}(\sigma)}{M_{j_i}(\sigma) - M_{j_{i-1}}(\sigma)}_{q_\mathfrak{o}^{-1}} \right) q_\mathfrak{o}^{\sum_{j \in [2d']} M_j(\sigma)(L_j(\sigma) - M_j(\sigma))(m_j - m_{j+1})}.$$

Proof. Recall the formula for $\alpha(\mu_\lambda, \nu; \mathfrak{o})$ in (2.3). Lemma 4.9 rewrites the product of Gaussian binomial coefficients in (2.3) as the product of Gaussian binomial coefficients in (4.10). That the power of $q_\mathfrak{o}$ in (2.3) equals the power of $q_\mathfrak{o}$ in (4.10) follows from the fact that $m_j = m_{j+1}$ if $L_j \neq L_j(\sigma)$ or $M_j \neq M_j(\sigma)$. □

4.3. **Partitioning the infinite summation into a finite number of parts.** The formula for $\zeta_{\mathfrak{f}_{2,d}(\mathfrak{o})}(s)$ in Proposition 4.3 is an infinite sum over the pairs $(\lambda, \nu) \in \mathcal{P}_d \times \mathcal{P}_{d'}$ that satisfy $\nu \leqslant \mu_\lambda$. We partition this infinite sum into a finite number of summations indexed by the elements of the set $\mathcal{W}_d$ from Definition 3.11. More precisely, the infinite number of summands indexed by the elements of $\mathcal{A}_d := \{(\lambda, \nu) \in \mathcal{P}_d \times \mathcal{P}_{d'} \mid \nu \leqslant \mu_\lambda\}$ are partitioned by the finitely many fibres of the following map ω.

Definition 4.11. Define the map

$$\omega : \mathcal{A}_d := \{(\lambda, \nu) \in \mathcal{P}_d \times \mathcal{P}_{d'} \mid \nu \leqslant \mu_\lambda\} \to \mathcal{W}_d : (\lambda, \nu) \mapsto \omega(\lambda, \nu) = (I, \sigma),$$

where $I = \{i \in [d-1] \mid \lambda_i > \lambda_{i+1}\}$ and $\sigma = \sigma_{\lambda,\nu}$ as in Definition 4.4.

Example 4.12. Let $d = 3$, $\lambda = (5, 4, 1)$, and $\nu = (6, 2, 2)$. The first component of $\omega(\lambda, \nu)$ is $I = \{1, 2\}$, as $\lambda_1 = 5 > \lambda_2 = 4$ and $\lambda_2 = 4 > \lambda_3 = 1$. By Example 4.5, $\sigma_{\lambda,\nu} = 451632$. Thus $\omega(\lambda, \nu) = (\{1, 2\}, 451632)$

Remark 4.13. The map ω is surjective by design of $\mathcal{W}_d$.

Remark 4.14. One motivation for defining the map ω from Definition 4.11 is that the Gaussian multinomial coefficients in (4.9) only depend on the image $\omega(\lambda, \nu) = (I, \sigma)$. This allows for the Gaussian multinomial coefficients to be pulled out of the summation over λ and ν, see (4.9).

Next, we show that the elements of the fibre $\omega^{-1}(I, \sigma)$ of ω are in bijection with the elements of the set $G_{I,\sigma}$ from Section 3.2.

Definition 4.15. For integer partitions $\lambda \in \mathcal{P}_d$ and $\nu \in \mathcal{P}_{d'}$, define

$$r_i = \lambda_i - \lambda_{i+1} \text{ for all } i \in [d-1], \qquad r_d = \lambda_d,$$
$$s_j = \nu_j - \nu_{j+1} \text{ for all } j \in [d'-1], \qquad s_{d'} = \nu_{d'}.$$

Example 4.16. Let $d = 3$, $\lambda = (5, 4, 1)$, and $\nu = (6, 2, 2)$. Then $r_1 = 5 - 4 = 1$, $r_2 = 4 - 1 = 3$, $r_3 = 1$, $s_1 = 6 - 2 = 4$, $s_2 = 2 - 2 = 0$, and $s_3 = 2$.

Recall the definition of the corresponding tuples v_i in Definition 3.6.

Lemma 4.17. *Let $\lambda \in \mathcal{P}_d$ and $\nu \in \mathcal{P}_{d'}$ be integer partitions with $\nu \leqslant \mu_\lambda$. Let $\sigma = \sigma_{\lambda,\nu}$ and write $\{m_i \mid i \in [2d']\} = \mu_\lambda \cup \nu$ with $m_i \geqslant m_{i+1}$ for all $i \in [2d'-1]$ as in Definition 2.4. Then*

$$m_i = \sum_{j=1}^{d} v_{\sigma(i),j} r_j + \sum_{j=1}^{d'} v_{\sigma(i),d+j} s_j.$$

Proof. Suppose that $i \in [2d']$ is such that $\sigma(i) \in [d']$. Then $v_{\sigma(i),j} = 0$ for $j \in [d]$ and

$$m_i = \nu_{\sigma(i)} = \sum_{j=\sigma(i)}^{d'} s_j = \sum_{j=1}^{d'} v_{\sigma(i),d+j} s_j.$$

Suppose that $i \in [2d']$ is such that $\sigma(i) \in d' + [d']$ and let $b^{-1}(\sigma(i)) = (l, m)$ where b is the map from Definition 3.3. Then $v_{\sigma(i),d+j} = 0$ for $j \in [d']$ and

$$m_i = \lambda_l + \lambda_m = \sum_{j=l}^{d} r_j + \sum_{j=m}^{d} r_j = \sum_{j=1}^{d} v_{\sigma(i),j} r_j. \qquad \square$$

Proposition 4.18. *Let $(I, \sigma) \in \mathcal{W}_d$. The map*

$$\omega^{-1}(I, \sigma) \to G_{I,\sigma} : (\lambda, \nu) \mapsto (\boldsymbol{r}, \boldsymbol{s}),$$

where $\boldsymbol{r} = (r_i)_{i\in[d']}$ and $\boldsymbol{s} = (s_i)_{i\in[d']}$ are as in Definition 4.15, is a bijection.

Proof. We first show that the map is well-defined. Let $\lambda \in \mathcal{P}_d$ and $\nu \in \mathcal{P}_{d'}$ be integer partitions with $\nu \leqslant \mu_\lambda$ and $\omega(\lambda, \nu) = (I, \sigma)$. Since λ and ν are integer partitions, we have that $r_i \geqslant 0$ and $s_j \geqslant 0$ for $i \in [d]$ and $j \in [d']$. The (in)equalities (3.3) and (3.4) follow from the definition of I in Definition 4.11 as $\{i \in [d-1] \mid \lambda_i > \lambda_{i+1}\}$. Let $\{m_i \mid i \in [2d']\} = \mu_\lambda \cup \nu$ with $m_i \geqslant m_{i+1}$ for all $i \in [2d'-1]$ as in Definition 2.4. Then (3.6) holds by Lemma 4.17. Moreover, from the definition of σ in Definition 4.11, it follows that $m_i > m_{i+1}$ when $i \in \mathrm{Asc}(\sigma)$, and therefore using Lemma 4.17, (3.5) also holds. Thus $(\boldsymbol{r}, \boldsymbol{s})$ is indeed an element of $G_{I,\sigma}$ and (4.18) is well defined.

Suppose that $(\boldsymbol{r}, \boldsymbol{s}) \in G_{I,\sigma}$. Let $\lambda_i = \sum_{j=i}^{d} r_j$ and $\nu_i = \sum_{j=i}^{d'} s_j$. Then $\lambda_i \geqslant \lambda_{i+1}$ and $\nu_j \geqslant \nu_{j+1}$ because $r_i, s_j \geqslant 0$ for all $i \in [d-1]$ and $j \in [d'-1]$. Let $\{m_i \mid i \in [2d']\} = \mu_\lambda \cup \nu$ with $m_i \geqslant m_{i+1}$ for all $i \in [2d'-1]$ as in Definition 2.4. Combining (3.5) and (3.6) with Lemma 4.17, we find that $\sigma_{\lambda,\nu} = \sigma$ and therefore $\nu \leqslant \mu_\lambda$ by Remark 4.6. Lastly, (3.3) and (3.4) imply that $\lambda_i > \lambda_{i+1}$ for all $i \in I$ and $\lambda_i = \lambda_{i+1}$ for all $i \in [d-1]\backslash I$. Thus (λ, ν) is an element of $\omega^{-1}(I, \sigma)$ that gets mapped to $(\boldsymbol{r}, \boldsymbol{s})$, proving that (4.18) is surjective. The injectivity is trivial. $\square$

4.4. **An explicit finite sum formula for $\zeta_{\mathfrak{f}_{2,d}(\mathfrak{o})}(s)$.** In Theorem 4.24, we reach the main result of Section 4. It writes the subalgebra zeta function $\zeta_{\mathfrak{f}_{2,d}(\mathfrak{o})}(s)$ as a finite sum indexed by the elements of $\mathcal{W}_d$, whose summands are a product of Gaussian multinomial coefficients and a substitution of a generating series of a set $G_{I,\sigma}$ from Section 3.2.

Definition 4.19. Let $(I, \sigma) \in \mathcal{W}_d$ and $\{j_i \mid i \in \{0\} \cup [r]\} := \mathrm{Asc}(\sigma) \cup \{0, 2d'\}$ with $j_i < j_{i+1}$ for all $i \in \{0\} \cup [r]$. The *product of Gaussian multinomial coefficients* $\mathrm{GMC}_{I,\sigma}$ *associated with* (I, σ) is

$$\mathrm{GMC}_{I,\sigma} = \binom{d}{I}_{q^{-1}} \left(\prod_{i\in[r]} \binom{L_{j_i}(\sigma) - M_{j_{i-1}}(\sigma)}{M_{j_i}(\sigma) - M_{j_{i-1}}(\sigma)}_{q^{-1}} \right) \in \mathbb{Z}[q^{-1}].$$

Example 4.20. Let $d = 3$ and $(I, \sigma) = (\{1, 2\}, 451623)$. Then

$$\mathrm{GMC}_{I,\sigma} = \binom{3}{\{1,2\}}_{q^{-1}} \binom{1-0}{0-0}_{q^{-1}} \binom{2-0}{1-0}_{q^{-1}} \binom{3-1}{2-1}_{q^{-1}} \binom{3-2}{3-2}_{q^{-1}}.$$

Let q and t be indeterminates and $\mathbf{X} = (X_i)_{i\in[d]}$ and $\mathbf{Y} = (Y_j)_{j\in[d']}$ be tuples of indeterminates.

Definition 4.21. Let $\sigma \in \mathcal{S}_{2d'}$. Define $x_i(\sigma), y_j(\sigma) \in \mathbb{Q}(q,t)$ to be

$$\begin{aligned} x_i(\sigma) &:= q^{\sum_{k\in[2d']} M_k(\sigma)(L_k(\sigma)-M_k(\sigma))(v_{\sigma(k),i}-v_{\sigma(k+1),i})} \cdot q^{i(d-i)} \cdot t^i \quad \text{for } i \in [d], \\ y_j(\sigma) &:= q^{\sum_{k\in[2d']} M_k(\sigma)(L_k(\sigma)-M_k(\sigma))(v_{\sigma(k),d+j}-v_{\sigma(k+1),d+j})} \cdot q^{jd}t^j \quad \text{for } j \in [d'], \end{aligned} \tag{4.4}$$

where $v_{\sigma(2d'+1),i} = 0$ when $i \in [d+d']$. The *numerical data map* χ_σ is

$$\chi_\sigma : \mathbb{Q}(\mathbf{X},\mathbf{Y}) \to \mathbb{Q}(q,t) : X_i \mapsto x_i(\sigma), Y_j \mapsto y_j(\sigma).$$

Example 4.22. Let $\sigma = 451623 \in \mathcal{S}_6$ and $i = 1$. Then $v_{\sigma(1),1} = v_{\sigma(2),1} = 1$ and $v_{\sigma(k),1} = 0$ for all $k \in \{3,4,5,6\}$. Thus

$$x_1(\sigma) = q^{0(1-0)(1-1)+0(2-0)(1-0)} \cdot q^{1(3-1)} \cdot t^1 = q^2 t.$$

Similarly, let $j = 1$, then $v_{\sigma(3),4} = 1$ and $v_{\sigma(k),4} = 0$ for all $k \in \{1,2,4,5,6\}$. Thus

$$y_1(\sigma) = q^{0(2-0)(0-1)+1(2-1)(1-0)} \cdot q^{1\cdot 3}t^1 = q^4 t.$$

Remark 4.23. Notice that, for each $i \in [d]$, the first exponent appearing in (4.4) can be rewritten as

$$\sum_{k\in[2d']} \left(M_k(\sigma)(L_k(\sigma)-M_k(\sigma)) - M_{k-1}(\sigma)(L_{k-1}(\sigma)-M_{k-1}(\sigma))\right) v_{\sigma(k),i}.$$

Given $i \in [d]$, $v_{\sigma(k),i}$ can only be non-zero if $\sigma(k) > d'$. In that case, $L_{k-1}(\sigma) < L_k(\sigma)$ and $M_{k-1}(\sigma) = M_k(\sigma)$ and therefore $M_k(\sigma)(L_k(\sigma)-M_k(\sigma)) - M_{k-1}(\sigma)(L_{k-1}(\sigma) - M_{k-1}(\sigma))$ is non-negative. It follows that (4.23) is non-negative as well.

Recall that $G_{I,\sigma}(\mathbf{X},\mathbf{Y})$ is the series enumerating the elements of $G_{I,\sigma}$. Let $\zeta_{\mathfrak{f}_{2,d}}(q,t)$ be the bivariate rational expression in $\mathbb{Q}(q,t)$ such that $\zeta_{\mathfrak{f}_{2,d}}(q_\mathfrak{o}, q_\mathfrak{o}^{-s})$ equals the $\mathfrak{p}$-adic zeta function $\zeta_{\mathfrak{f}_{2,d}(\mathfrak{o})}(s)$ for all cDVR $\mathfrak{o}$. The following is the main result of Section 4.

Theorem 4.24. *For all $d \in \mathbb{N}_{\geqslant 2}$,*

$$\zeta_{\mathfrak{f}_{2,d}}(q,t) = \sum_{(I,\sigma)\in\mathcal{W}_d} \mathrm{GMC}_{I,\sigma}\, \chi_\sigma(G_{I,\sigma}(\mathbf{X},\mathbf{Y})). \tag{4.5}$$

Proof. By Proposition 4.3, we may write

$$\zeta_{\mathfrak{f}_{2,d}(\mathfrak{o})}(s) = \sum_{(I,\sigma)\in\mathcal{W}_d} \sum_{\lambda\in\mathcal{P}_d} \sum_{\substack{\nu\in\mathcal{P}_{d'},\nu\leqslant\mu_\lambda,\\ \omega(\lambda,\nu)=(I,\sigma)}} \alpha\left(\lambda_1^{(n)}, \lambda; \mathfrak{o}\right) \alpha(\mu_\lambda, \nu; \mathfrak{o}) q_\mathfrak{o}^{-s|\lambda|} q_\mathfrak{o}^{(d-s)|\nu|}. \tag{4.6}$$

Corollary 2.7 with $n = d$ tells us that

$$\alpha\left(\lambda_1^{(d)}, \lambda; \mathfrak{o}\right) = \binom{d}{I}_{q_\mathfrak{o}^{-1}} \prod_{i=1}^{d} q_\mathfrak{o}^{i(d-i)(\lambda_i-\lambda_{i+1})}. \tag{4.7}$$

For integer partitions $\lambda \in \mathcal{P}_d$ and $\nu \in \mathcal{P}_{d'}$ with $\nu \leqslant \mu_\lambda$, let

$$D_{\lambda,\nu} = q^{\sum_{j\in[2d']} M_j(\sigma)(L_j(\sigma)-M_j(\sigma))(m_j-m_{j+1})} q^{\sum_{i=1}^d i(d-i)(\lambda_i-\lambda_{i+1})} t^{|\lambda|} (q^d t)^{|\nu|}, \tag{4.8}$$

where $\sigma = \sigma_{\lambda,\nu}$ and $\{m_i \mid i \in [2d']\} = \mu_\lambda \cup \nu$ with $m_i \geqslant m_{i+1}$ for all $i \in [2d'-1]$ as in Definition 2.4. Using (4.7) and Proposition 4.10 in (4.6) results in

$$\zeta_{\mathfrak{f}_{2,d}}(q,t) = \sum_{(I,\sigma)\in\mathcal{W}_d} \mathrm{GMC}_{I,\sigma} \sum_{\lambda\in\mathcal{P}_d} \sum_{\substack{\nu\in\mathcal{P}_{d'},\nu\leqslant\mu_\lambda,\\ \omega(\lambda,\nu)=(I,\sigma)}} D_{\lambda,\nu}. \tag{4.9}$$

It now suffices to show that for each $(I,\sigma) \in \mathcal{W}_d$,

$$\sum_{\lambda\in\mathcal{P}_d} \sum_{\substack{\nu\in\mathcal{P}_{d'},\nu\leqslant\mu_\lambda,\\ \omega(\lambda,\nu)=(I,\sigma)}} D_{\lambda,\nu} = \chi_\sigma(G_{I,\sigma}(\mathbf{X},\mathbf{Y})). \tag{4.10}$$

By Proposition 4.18, the summands on the left-hand side of (4.10) are in bijection with the elements of $G_{I,\sigma}$ and it suffices to show that

$$D_{\lambda,\nu} = \left(\prod_{i=1}^{d} x_i(\sigma)^{r_i}\right)\left(\prod_{j=1}^{d'} y_j(\sigma)^{s_j}\right), \tag{4.11}$$

where the r_i and s_j are as in Definition 4.15. The $D_{\lambda,\nu}$ in (4.8) are written as a product of four powers of q, which we analyze in turn.

(1) The first power of q has exponent $\sum_{j\in[2d']} M_j(\sigma)(L_j(\sigma)-M_j(\sigma))(m_j-m_{j+1})$. It follows from Lemma 4.17 that

$$m_k - m_{k+1} = \sum_{i=1}^{d}(v_{\sigma(k),i} - v_{\sigma(k+1),i})r_i + \sum_{j=1}^{d'}(v_{\sigma(k),d+j} - v_{\sigma(k+1),d+j})s_j, \tag{4.12}$$

for all $k \in [2d']$, where $v_{\sigma(2d'+1),i} = 0$ for each $i \in [d+d']$. We may thus rewrite this first power as

$$\left(\prod_{i=1}^{d}\left(q^{\sum_{k\in[2d']} M_k(\sigma)(L_k(\sigma)-M_k(\sigma))(v_{\sigma(k),i}-v_{\sigma(k+1),i})}\right)^{r_i}\right) \cdot \textit{ nonumber} \tag{4.13}$$

$$\left(\prod_{j=1}^{d'}\left(q^{\sum_{k\in[2d']} M_k(\sigma)(L_k(\sigma)-M_k(\sigma))(v_{\sigma(k),d+j}-v_{\sigma(k+1),d+j})}\right)^{s_j}\right).$$

(2) The second power q is easily rewritten as follows,

$$q^{\sum_{i=1}^{d} i(d-i)(\lambda_i-\lambda_{i+1})} = \left(\prod_{i=1}^{d}\left(q^{i(d-i)}\right)^{r_i}\right)\left(\prod_{j=1}^{d'}(1)^{s_j}\right). \tag{4.14}$$

(3) The third and fourth powers of q can be written as

$$q^{-s|\lambda|}q^{(d-s)|\nu|} = \left(\prod_{i=1}^{d}\left(t^i\right)^{r_i}\right)\left(\prod_{j=1}^{d'}\left(q^{jd}t^j\right)^{s_j}\right). \tag{4.15}$$

The product of (4.13) with the right-hand sides of (4.14) and (4.15) indeed results in (4.11). □

5. Overlap type zeta functions, functional equation and pole at zero

In this section, we present results on the $\mathfrak{p}$-adic zeta function $\zeta_{\mathfrak{f}_{2,d}(\mathfrak{o})}(s)$. In Section 5.1 we introduce the *overlap type zeta functions* $\zeta^{w}_{\mathfrak{f}_{2,d}(\mathfrak{o})}(s)$ for each Dyck word $w \in \mathcal{D}_{2d'}$, which are special summands of $\zeta_{\mathfrak{f}_{2,d}(\mathfrak{o})}(s)$. Special attention goes to one overlap type zeta function called the *no-overlap zeta function*. Informally and purely heuristically speaking, it enumerates "most" of the subalgebras of $\mathfrak{f}_{2,d}(\mathfrak{o})$. Theorem 5.9 establishes a functional equation for the no-overlap zeta function, while Theorem 5.11 proves that it has a simple pole at zero. In Section 5.5 we prove that the $\mathfrak{p}$-adic zeta function $\zeta_{\mathfrak{f}_{2,d}(\mathfrak{o})}(s)$ has a simple pole at zero as well, confirming a conjecture of Rossmann in the relevant cases; see Theorem 5.13.

5.1. **Overlap types and overlap zeta functions.** We define the overlap type $w(\mathfrak{h})$ of a subalgebra $\mathfrak{h} \leqslant \mathfrak{f}_{2,d}(\mathfrak{o})$ of finite index and define an overlap type zeta function $\zeta^{w}_{\mathfrak{f}_{2,d}(\mathfrak{o})}(s)$ for each overlap type w, which enumerates the subalgebras of $\mathfrak{f}_{2,d}(\mathfrak{o})$ with that overlap type. Afterwards, we slightly adapt Theorem 4.24 to a formula for each overlap type zeta function $\zeta^{w}_{\mathfrak{f}_{2,d}(\mathfrak{o})}(s)$.

Recall from Section 3.6 that $\mathcal{D}_{2d'}$ denotes the set of Dyck words of length $2d'$ and w_σ is the Dyck word associated with σ. Recall, moreover, the permutation $\sigma_{\lambda,\nu}$ associated with (λ,ν) from Definition 4.4.

Definition 5.1. Let $\mathfrak{h}$ be a subalgebra of $\mathfrak{f}_{2,d}(\mathfrak{o})$ of finite index. Let λ be the elementary divisor type of $\mathfrak{h}/[\mathfrak{f}_{2,d}(\mathfrak{o}),\mathfrak{f}_{2,d}(\mathfrak{o})]$ in $\mathfrak{f}_{2,d}(\mathfrak{o})/[\mathfrak{f}_{2,d}(\mathfrak{o}),\mathfrak{f}_{2,d}(\mathfrak{o})]$ as in Definition 2.8. Let ν be the elementary divisor type of $\mathfrak{h}\cap[\mathfrak{f}_{2,d}(\mathfrak{o}),\mathfrak{f}_{2,d}(\mathfrak{o})]$ in $[\mathfrak{f}_{2,d}(\mathfrak{o}),\mathfrak{f}_{2,d}(\mathfrak{o})]$. The *overlap type* $w(\mathfrak{h})$ of $\mathfrak{h}$ is the Dyck word $w_{\sigma_{\lambda,\nu}}\in\mathcal{D}_{2d'}$.

We say that $\mathfrak{h}$ has *no overlap* if $w_{\sigma_{\lambda,\nu}}$ is the trivial Dyck word $0^{d'}1^{d'}$. Equivalently, $\mathfrak{h}$ has no overlap if and only if $\mu_1\geqslant\cdots\geqslant\mu_{d'}\geqslant\nu_1\geqslant\cdots\geqslant\nu_{d'}$ where $(\mu_j)_{j\in[d']}:=\mu_\lambda$ as in Definition 4.1. This is equivalent to the valuations of the elementary divisors of $[\mathfrak{f}_{2,d}(\mathfrak{o}),\mathfrak{f}_{2,d}(\mathfrak{o})]/(\mathfrak{h}\cap[\mathfrak{f}_{2,d}(\mathfrak{o}),\mathfrak{f}_{2,d}(\mathfrak{o})])$ all being less than or equal to all the valuations of elementary divisors of $[\mathfrak{f}_{2,d}(\mathfrak{o}),\mathfrak{f}_{2,d}(\mathfrak{o})]/[\mathfrak{h},\mathfrak{h}]$. Otherwise, we say that $\mathfrak{h}$ has *overlap.*

Definition 5.2. Let $w\in\mathcal{D}_{2d'}$. The *overlap type w zeta function* is defined as

$$\zeta^{w}_{\mathfrak{f}_{2,d}(\mathfrak{o})}(s):=\sum_{\mathfrak{h}\leqslant\mathfrak{f}_{2,d}(\mathfrak{o}),w(\mathfrak{h})=w}|\mathfrak{f}_{2,d}(\mathfrak{o}):\mathfrak{h}|^{-s}.$$

In particular, the *no-overlap zeta function* is defined as

$$\zeta^{\text{n.o.}}_{\mathfrak{f}_{2,d}(\mathfrak{o})}(s):=\zeta^{0^{d'}1^{d'}}_{\mathfrak{f}_{2,d}(\mathfrak{o})}(s)=\sum_{\mathfrak{h}\leqslant\mathfrak{f}_{2,d}(\mathfrak{o}),w(\mathfrak{h})=0^{d'}1^{d'}}|\mathfrak{f}_{2,d}(\mathfrak{o}):\mathfrak{h}|^{-s}.$$

Obviously $\zeta_{\mathfrak{f}_{2,d}(\mathfrak{o})}(s)=\sum_{w\in\mathcal{D}_{2d'}}\zeta^{w}_{\mathfrak{f}_{2,d}(\mathfrak{o})}(s)$. These functions may be compared with the summands $D_{w,\rho}(q,t)$ in [5, Def. 4.18].

Let $\zeta^{w}_{\mathfrak{f}_{2,d}}(q,t)$ be the bivariate rational expression in $\mathbb{Q}(q,t)$ such that $\zeta^{w}_{\mathfrak{f}_{2,d}}(q_\mathfrak{o},q_\mathfrak{o}^{-s})$ equals $\zeta^{w}_{\mathfrak{f}_{2,d}(\mathfrak{o})}(s)$ for all cDVR $\mathfrak{o}$. Theorem 4.24 is straightforwardly adapted to obtain a formula for the overlap type zeta functions $\zeta^{w}_{\mathfrak{f}_{2,d}(\mathfrak{o})}(s)$ as follows.

Theorem 5.3. *For all $d\in\mathbb{N}_{\geqslant 2}$ and $w\in\mathcal{D}_{2d'}$. Then*

$$\zeta^{w}_{\mathfrak{f}_{2,d}}(q,t)=\sum_{\substack{(I,\sigma)\in\mathcal{W}_d\\ w_\sigma=w}}\mathrm{GMC}_{I,\sigma}\,\chi_\sigma(G_{I,\sigma}(\mathbf{X},\mathbf{Y})).\tag{5.1}$$

Proof. In the statement and proof of Proposition 4.3, the summation can be restricted to the integer partitions $\lambda\in\mathcal{P}_d$ and $\nu\in\mathcal{P}_{d'}$ with $\nu\leqslant\mu_\lambda$ and $w_{\sigma_{\lambda,\nu}}=w$:

$$\zeta^{w}_{\mathfrak{f}_{2,d}(\mathfrak{o})}(s)=\sum_{\substack{\lambda\in\mathcal{P}_d,\ \nu\in\mathcal{P}_{d'},\\ \nu\leqslant\mu_\lambda,\ w_{\sigma_{\lambda,\nu}}=w}}\alpha\left(\lambda_1^{(n)},\lambda;\mathfrak{o}\right)\alpha(\mu_\lambda,\nu;\mathfrak{o})q_\mathfrak{o}^{-s|\lambda|}q_\mathfrak{o}^{(d-s)|\nu|}.$$

Similarly, in the statement and proof of Theorem 4.24, the summation can be restricted to pairs $(I,\sigma)\in\mathcal{W}_d$ with $w_\sigma=w$, resulting in (5.1). □

5.2. **An alternative formula for the no-overlap zeta function.** We simplify the formula for $\zeta^{w}_{\mathfrak{f}_{2,d}(\mathfrak{o})}(s)$ in Theorem 5.3 in the case when $w=0^{d'}1^{d'}$, i.e. for the no-overlap zeta function $\zeta^{\text{n.o.}}_{\mathfrak{f}_{2,d}(\mathfrak{o})}(s)$. We start by establishing how the products of Gaussian binomial coefficients $\mathrm{GMC}_{I,\sigma}$ simplify when $w_\sigma=0^{d'}1^{d'}$.

Lemma 5.4. *Suppose that $(I,\sigma)\in\mathcal{W}_d$ with $w_\sigma=0^{d'}1^{d'}$. With*

$$J_\sigma:=\{j\in[d'-1]\mid\sigma^{-1}(j)<\sigma^{-1}(j+1)\},$$

we have

$$\mathrm{GMC}_{I,\sigma}=\binom{d}{I}_{q^{-1}}\binom{d'}{J_\sigma}_{q^{-1}}.\tag{5.2}$$

Proof. If $w_\sigma = 0^{d'}1^{d'}$, then $M_0(\sigma) = \cdots = M_{d'}(\sigma) = 0$ and therefore

$$\binom{L_{j_i}(\sigma) - M_{j_{i-1}}(\sigma)}{M_{j_i}(\sigma) - M_{j_{i-1}}(\sigma)}_{q^{-1}} = 1$$

for all $j_i \in \mathrm{Asc}(\sigma) \cap [d']$. Moreover, $L_{d'+j}(\sigma) = d'$ and $M_{d'+j}(\sigma) = j$ for all $j \in [d']$. Let $j_1 > \cdots > j_r$ be such that $\mathrm{Asc}(\sigma) \cap (d' + [d'-1]) = d' + \{j_i \mid i \in [r]\}$. Because $\sigma \in \mathcal{S}_{2d'}$, it follows that $\sigma(d' + j_{i-1} + k) = j_i + 1 - k$ for all $i \in [r]$ and $k \in [j_i - j_{i-1} - 1]$ where $j_0 = 0$. Therefore $\mathrm{Asc}(\sigma) \cap (d' + [d'-1]) = d' + J_\sigma$. Thus

$$\prod_{j_i \in \mathrm{Asc}(\sigma) \cap (d' + [d'])} \binom{L_{j_i}(\sigma) - M_{j_{i-1}}(\sigma)}{M_{j_i}(\sigma) - M_{j_{i-1}}(\sigma)}_{q^{-1}} = \prod_{j_i \in J_\sigma} \binom{d' - j_{i-1}}{j_i - j_{i-1}}_{q^{-1}} = \binom{d'}{J_\sigma}_{q^{-1}} \qquad \square$$

Next, we establish what the numerical data map χ_σ simplifies to when $w_\sigma = 0^{d'}1^{d'}$.

Definition 5.5. The *no-overlap numerical data map* $\chi_{\text{n.o.}}$ is

$$\chi_{\text{n.o.}} : \mathbb{Q}(\mathbf{X}, \mathbf{Y}) \to \mathbb{Q}(q,t) : X_i \mapsto q^{i(d-i)}t^i, Y_j \mapsto q^{dj + j(d'-j)}t^j.$$

Example 5.6. If $d = 3$, then $\chi_{\text{n.o.}}(X_1) = q^{1(3-1)}t^1 = q^2t$ and $\chi_{\text{n.o.}}(Y_1) = q^{3\cdot 1 + 1(3-1)}t^1 = q^5t$.

Remark 5.7. Let $\sigma \in \mathcal{S}_{2d'}$ be such that $w_\sigma = 0^{d'}1^{d'}$. Then the numerical data map χ_σ from Definition 4.21 simplifies to $\chi_{\text{n.o.}}$.

The following theorem provides an alternative formula for the no-overlap zeta function $\zeta^{\text{n.o.}}_{\mathfrak{f}_{2,d}(\mathfrak{o})}(s)$. It is a lot less complicated than the general formula for the overlap type w zeta function $\zeta^{w}_{\mathfrak{f}_{2,d}(\mathfrak{o})}(s)$ in Theorem 5.3 because it has fewer summands and the summands are simpler. Recall the sets $H_{I,J}$ from Section 3.5.

Theorem 5.8. *For all $d \in \mathbb{N}_{\geqslant 2}$,*

$$\zeta^{\text{n.o.}}_{\mathfrak{f}_{2,d}}(q,t) = \sum_{I \subseteq [d-1], J \subseteq [d'-1]} \binom{d}{I}_{q^{-1}} \binom{d'}{J}_{q^{-1}} \chi_{\text{n.o.}}(H_{I,J}(\mathbf{X}, \mathbf{Y})). \tag{5.3}$$

Proof. Using (5.2) and Remark 5.7 in (5.1), results in

$$\zeta^{\text{n.o.}}_{\mathfrak{f}_{2,d}}(q,t) = \sum_{\substack{(I,\sigma) \in \mathcal{W}_d, \\ w_\sigma = 0^{d'}1^{d'}}} \binom{d}{I}_{q^{-1}} \binom{d'}{J_\sigma}_{q^{-1}} \chi_{\text{n.o.}}(G_{I,\sigma}(\mathbf{X}, \mathbf{Y})).$$

Now using Proposition 3.40, we find (5.3). $\square$

5.3. **A functional equation for the no-overlap zeta function.** [17, Thm A] asserts in particular that $\zeta_{\mathfrak{f}_{2,d}}(q,t)$ satisfies the functional equation

$$\zeta_{\mathfrak{f}_{2,d}}(q^{-1}, t^{-1}) = (-1)^D q^{\binom{D}{2}} t^D \zeta_{\mathfrak{f}_{2,d}}(q,t),$$

where $D := d + d' = \binom{d+1}{2}$, the $\mathbb{Z}$-rank of $\mathfrak{f}_{2,d}$. The following theorem determines that the no-overlap zeta function $\zeta^{\text{n.o.}}_{\mathfrak{f}_{2,d}}(q,t)$ satisfies the same functional equation.

Theorem 5.9. *For all $d \in \mathbb{N}_{\geqslant 2}$, the no-overlap zeta function $\zeta^{\text{n.o.}}_{\mathfrak{f}_{2,d}(\mathfrak{o})}(s)$ satisfies the functional equation*

$$\zeta^{\text{n.o.}}_{\mathfrak{f}_{2,d}}(q^{-1}, t^{-1}) = (-1)^D q^{\binom{D}{2}} t^D \zeta^{\text{n.o.}}_{\mathfrak{f}_{2,d}}(q,t),$$

where $D := d + d' = \binom{d+1}{2}$ is the $\mathbb{Z}$-rank of $\mathfrak{f}_{2,d}$.

Proof. This proof follows the proof of [18, Thm. 2.15] which refers to [17, Sec. 2 and 3]. We start from the formula for $\zeta^{\text{n.o.}}_{\mathfrak{f}_{2,d}}(q,t)$ stated in Theorem 5.8. Using the identity (2.1) for the Gaussian multinomial coefficients, we find

$$\zeta^{\text{n.o.}}_{\mathfrak{f}_{2,d}}(q,t) = \sum_{\substack{I\subseteq[d-1]\\ J\subseteq[d'-1]}} \left(\sum_{\substack{w\in S_d,\\ \mathrm{Des}(w)\subseteq I}} q^{-\ell(w)}\right)\left(\sum_{\substack{v\in S_{d'},\\ \mathrm{Des}(v)\subseteq J}} q^{-\ell(v)}\right)\chi_{\text{n.o.}}(H_{I,J}(\mathbf{X},\mathbf{Y})).$$

Reordering the summations, this becomes

$$\zeta^{\text{n.o.}}_{\mathfrak{f}_{2,d}}(q,t) = \sum_{w\in S_d} q^{-\ell(w)} \sum_{v\in S_{d'}} q^{-\ell(v)} \sum_{\substack{\mathrm{Des}(w)\subseteq I\subseteq[d-1]\\ \mathrm{Des}(v)\supseteq J\subseteq[d'-1]}} \chi_{\text{n.o.}}(H_{I,J}(\mathbf{X},\mathbf{Y})).$$

Inverting q on both sides and using Proposition 3.36, we find that $\zeta^{\text{n.o.}}_{\mathfrak{f}_{2,d}}(q^{-1},t^{-1})$ equals

$$(-1)^{d+d'} \sum_{w\in S_d} q^{\ell(w)} \sum_{v\in S_{d'}} q^{\ell(v)} \sum_{\substack{[d-1]\setminus \mathrm{Des}(w)\subseteq I\subseteq[d-1]\\ [d'-1]\setminus \mathrm{Des}(v)\subseteq J\subseteq[d'-1]}} \chi_{\text{n.o.}}(X_dY_{d'}H_{I,J}(\mathbf{X},\mathbf{Y})).$$

Using the two equations in (2.1), $\zeta^{\text{n.o.}}_{\mathfrak{f}_{2,d}}(q^{-1},t^{-1})$ becomes

$$(-1)^{d+d'} \sum_{w\in S_d} q^{d'-\ell(ww_0)} \sum_{v\in S_{d'}} q^{\binom{d'}{2}-\ell(vv_0)} \sum_{\substack{\mathrm{Des}(ww_0)\subseteq I\subseteq[d-1]\\ \mathrm{Des}(vv_0)\subseteq J\subseteq[d'-1]}} \chi_{\text{n.o.}}(X_dY_{d'}H_{I,J}(\mathbf{X},\mathbf{Y})).$$

Changing the order of summation again results in

$$\begin{aligned}\zeta^{\text{n.o.}}_{\mathfrak{f}_{2,d}}(q^{-1},t^{-1}) =&(-1)^{d+d'}q^{d'+\binom{d'}{2}} \sum_{\substack{I\subseteq[d-1]\\ J\subseteq[d'-1]}} \left(\sum_{\substack{w\in S_d,\\ \mathrm{Des}(ww_0)\subseteq I}} q^{-\ell(ww_0)}\right)\\ &\left(\sum_{\substack{v\in S_{d'},\\ \mathrm{Des}(vv_0)\subseteq J}} q^{-\ell(vv_0)}\right)\chi_{\text{n.o.}}(X_dY_{d'}H_{I,J}(\mathbf{X},\mathbf{Y})).\end{aligned}$$

Using (2.1) and $\chi_{\text{n.o.}}(X_dY_{d'}) = q^{dd'}t^{d+d'}$ yields

$$\zeta^{\text{n.o.}}_{\mathfrak{f}_{2,d}}(q^{-1},t^{-1}) = (-1)^{d+d'}q^{d'+\binom{d'}{2}+dd'} \sum_{\substack{I\subseteq[d-1]\\ J\subseteq[d'-1]}} \binom{d}{I}_{q^{-1}}\binom{d'}{J}_{q^{-1}}\chi_{\text{n.o.}}(H_{I,J}(\mathbf{X},\mathbf{Y})).$$

Lastly, using $d'+\binom{d'}{2}+dd' = D$ and Theorem 5.8 yields (5.9). $\square$

In light of results such as [5, Prop. 4.19], one might expect that the functional equation established in Theorem 5.9 might hold for all $\zeta^{w}_{\mathfrak{f}_{2,d}}(q,t)$ where $w\in\mathcal{D}_{2d'}$, not just for $w = 0^{d'}1^{d'}$. Our explicit calculations (see Section 7) find that this indeed holds for $d\leqslant 4$.

Conjecture 5.10. *For all $d\in\mathbb{N}_{\geqslant 2}$ and w any Dyck word of length $2d'$, the overlap type zeta function $\zeta^{w}_{\mathfrak{f}_{2,d}(\mathfrak{o})}(s)$ satisfies the functional equation*

$$\zeta^{w}_{\mathfrak{f}_{2,d}}(q^{-1},t^{-1}) = (-1)^Dq^{\binom{D}{2}}t^D\zeta^{w}_{\mathfrak{f}_{2,d}}(q,t),$$

where $D := d+d' = \binom{d+1}{2}$ is the $\mathbb{Z}$-rank of $\mathfrak{f}_{2,d}$.

5.4. **The simple pole at zero of the no-overlap zeta function.** Next, we study the behaviour of $\zeta^{\mathrm{n.o.}}_{\mathfrak{f}_{2,d}(\mathfrak{o})}(s)$ at $s=0$.

Theorem 5.11. *The no-overlap zeta function* $\zeta^{\mathrm{n.o.}}_{\mathfrak{f}_{2,d}(\mathfrak{o})}(s)$ *has a simple pole at* $s=0$ *for all but finitely many* $q_{\mathfrak{o}}$.

Proof. We start from the formula for $\zeta^{\mathrm{n.o.}}_{\mathfrak{f}_{2,d}}(q,t)$ stated in Theorem 5.8. Let $I \subseteq [d-1]$ and $J \subseteq [d'-1]$. Proposition 3.34 implies that $H_{I,J}(\mathbf{X},\mathbf{Y}) = I_{E_{\mathrm{n.o.}},A_{I,J},C_{I,J}}(\mathbf{X},\mathbf{Y},1)$. Let $\Gamma_{I,J} = \{K_u \mid u \in U_{I,J}\}$ be a family of simplicial monoids, satisfying the conditions in Proposition 2.29 with $E = E_{\mathrm{n.o.}}$, $A = A_{I,J}$, and $C = C_{I,J}$. Using that $H_{I,J}(\mathbf{X},\mathbf{Y}) = I_{E_{\mathrm{n.o.}},A_{I,J},C_{I,J}}(\mathbf{X},\mathbf{Y},1)$ and $\bigcup_{u\in U_{I,J}} \overline{K_u} = I_{E_{\mathrm{n.o.}},A_{I,J},C_{I,J}}$ in Theorem 5.8 results in

$$\zeta^{\mathrm{n.o.}}_{\mathfrak{f}_{2,d}}(q,t) = \sum_{I\subseteq[d-1],J\subseteq[d'-1]} \binom{d}{I}_{q^{-1}} \binom{d'}{J}_{q^{-1}} \sum_{u\in U_{I,J}} \chi_{\mathrm{n.o.}}(\overline{K_u}(\mathbf{X},\mathbf{Y},1)). \tag{5.4}$$

By Theorem 2.19,

$$\overline{K_u}(\mathbf{Z}) = \frac{\sum_{\beta\in D_{\overline{K_u}}} \mathbf{Z}^{\beta}}{\prod_{i=1}^{r}(1-\mathbf{Z}^{\alpha_i})}, \tag{5.5}$$

where $\alpha_1, \ldots, \alpha_r$ are quasigenerators of K_u, $D_{\overline{K_u}}$ is defined in (2.6), and $\mathbf{Z} = (\mathbf{X},\mathbf{Y},1)$. Therefore $\chi_{\mathrm{n.o.}}(\overline{K_u}(\mathbf{Z}))|_{q\to q_{\mathfrak{o}},t\to q_{\mathfrak{o}}^{-s}}$ has a pole at $s=0$ if and only if there is a quasigenerator γ of K_u such that $\chi_{\mathrm{n.o.}}(\mathbf{Z}^\gamma) = \chi_{\mathrm{n.o.}}(X_1^{\gamma_1}\ldots Y_{d'}^{\gamma_{d+d'}})$ is a power of t. Looking at Definition 5.5, this means that the monomial $\mathbf{Z}^\gamma$ has to have degree zero in the variables $X_1,\ldots,X_{d-1}$, $Y_1,\ldots, Y_{d'}$. Thus the support of γ has to be contained in $\{d, d+d'+1\}$. We show that there is a K_u that has a quasigenerator whose support is contained in $\{d, d+d'+1\}$.

For each $i \in [d+d'+1]$, let $\delta_i \in \mathbb{N}_0^{d+d'+1}$ be ith unit basis vector. As discussed in Section 3.4, $\delta_d + 2\delta_{d+d'+1}$ is a completely fundamental element of $E_{\mathrm{n.o.}}$ and therefore by Proposition 2.29, it is a quasigenerator for some of the K_u. There cannot be more than one quasigenerator of $\mathcal{C}_{E_{\mathrm{n.o.}}}$ with the same support and thus the multiplicity of the pole at $s=0$ of $\chi_{\mathrm{n.o.}}(\overline{K_u}(\mathbf{Z}))|_{q\to q_{\mathfrak{o}},t\to q_{\mathfrak{o}}^{-s}}$ is at most one. The order of a pole of a sum is at most the maximal order of the poles of the summands. Therefore by (5.4), $\zeta^{\mathrm{n.o.}}_{\mathfrak{f}_{2,d}(\mathfrak{o})}(s)$ can have at most a simple pole at $s=0$ (the Gaussian binomial coefficients do not depend on s and therefore they have no poles or zeros).

It remains to show that the residues of the summands in (5.4) do not cancel each other out except for possibly finitely many $q_{\mathfrak{o}}$. To that end, it suffices to show that

$$\sum_{I\subseteq[d-1],J\subseteq[d'-1]} \binom{d}{I}_{q^{-1}} \binom{d'}{J}_{q^{-1}} \lim_{s\to 0} s \sum_{u\in U_{I,J}} \chi_{\mathrm{n.o.}}(\overline{K_u}(\mathbf{Z}))\big|_{t\to q^{-s}} \tag{5.6}$$

is a non-zero rational expression in q multiplied by $\log q$. The Gaussian multinomial coefficients are polynomials in q^{-1} with non-negative coefficients. By the reasoning earlier in this proof, $\chi_{\mathrm{n.o.}}(\overline{K_u}(\mathbf{Z}))\big|_{q\to q_{\mathfrak{o}},t\to q_{\mathfrak{o}}^{-s}}$ can have at most a simple pole at $s=0$. If it has no pole, then $\lim_{s\to0} s\sum_{u\in U_{I,J}} \chi_{\mathrm{n.o.}}(\overline{K_u}(\mathbf{Z}))\big|_{t\to q^{-s}}$ is zero. Otherwise, (5.5) shows that the numerator of $\chi_{\mathrm{n.o.}}(\overline{K_u}(\mathbf{Z}))$ is a polynomial in q and t with non-negative coefficients. It also shows that the denominator of $\chi_{\mathrm{n.o.}}(\overline{K_u}(\mathbf{Z}))$ is a product of $\dim K_u$ polynomials of the form $(1-q^a t^b)$ with $a,b\in\mathbb{Z}$. Because we assume that $\chi_{\mathrm{n.o.}}(\overline{K_u}(\mathbf{Z}))\big|_{q\to q_{\mathfrak{o}},t\to q_{\mathfrak{o}}^{-s}}$ has a simple pole at $s=0$, exactly one of these factors has $a=0$. Therefore multiplying the denominator by s^{-1} and taking the limit $s\to0$ results in $\log q$ multiplied by a product of polynomials of the form $(1-q^a)$. Thus in this case, $\lim_{s\to0} s\sum_{u\in U_{I,J}} \chi_{\mathrm{n.o.}}(\overline{K_u}(\mathbf{Z}))\big|_{t\to q^{-s}}$ is a non-zero rational expression in q

multiplied by $\log q$. Thus we find that (5.6) is indeed a rational expression in q multiplied by $\log q$. To show that it is non-zero, notice that when evaluated at $q = \frac{1}{2}$, both the Gaussian multinomial coefficients, the numerators and the factors $(1 - q^a)$ in de denominators of the $\lim_{s\to 0} s \sum_{u\in U_{I,J}} \chi_{\text{n.o.}}(\overline{K_u}(\mathbf{Z}))|_{t\to q^{-s}}$ are non-negative numbers, while $\log 1/2$ is a negative number. Therefore the summands in (5.6) all have the same sign when evaluated in $q = 1/2$ and cannot completely cancel out. □

5.5. **The simple pole at zero of the overlap and subalgebra zeta functions.** We study the behaviour of $\zeta_{\mathfrak{f}_{2,d}(\mathfrak{o})}(s)$ at $s = 0$ by first looking at the behaviour of $\zeta_{\mathfrak{f}_{2,d}(\mathfrak{o})}(s) - \zeta^{\text{n.o.}}_{\mathfrak{f}_{2,d}(\mathfrak{o})}(s)$ at $s = 0$.

Theorem 5.12. *The rational function $\zeta_{\mathfrak{f}_{2,d}(\mathfrak{o})}(s) - \zeta^{\text{n.o.}}_{\mathfrak{f}_{2,d}(\mathfrak{o})}(s)$ does not have a pole at $s = 0$.*

Proof. Since $\zeta_{\mathfrak{f}_{2,d}(\mathfrak{o})}(s) - \zeta^{\text{n.o.}}_{\mathfrak{f}_{2,d}(\mathfrak{o})}(s) = \sum_{w\in\mathcal{D}_{2d'}, w\neq 0^{d'}1^{d'}} \zeta^{w}_{\mathfrak{f}_{2,d}(\mathfrak{o})}(s)$ it follows from Theorem 5.3 that

$$\zeta_{\mathfrak{f}_{2,d}}(q,t) - \zeta^{\text{n.o.}}_{\mathfrak{f}_{2,d}}(q,t) = \sum_{(I,\sigma)\in\mathcal{W}_d, w_\sigma\neq 0^{d'}1^{d'}} \mathrm{GMC}_{I,\sigma}\, \chi_\sigma(G_{I,\sigma}(\mathbf{X},\mathbf{Y})).$$

Therefore it suffices to show that for each pair $(I,\sigma) \in \mathcal{W}_d$ with $w_\sigma \neq 0^{d'}1^{d'}$, the rational function $\chi_\sigma(G_{I,\sigma}(\mathbf{X},\mathbf{Y}))|_{q\to q_\mathfrak{o}, t\to q_\mathfrak{o}^{-s}}$ has no pole at $s = 0$. Recall from Proposition 3.22 that $G_{I,\sigma}$ can be seen as the projection of $I_{E_\sigma, A_{I,\sigma}, C_{I,\sigma}}$ on the first $d + d'$ coordinates, thus $G_{I,\sigma}(\mathbf{X},\mathbf{Y}) = I_{E_\sigma, A_{I,\sigma}, C_{I,\sigma}}(\mathbf{X},\mathbf{Y},\mathbf{1})$. Let $\Gamma_{I,\sigma} = \{K_u \mid u \in U_{I,\sigma}\}$ be as in Definition 3.23. Using that $\bigcup_{u\in U_{I,\sigma}} \overline{K_u} = I_{E_\sigma, A_{I,\sigma}, C_{I,\sigma}}$, we find

$$\chi_\sigma(G_{I,\sigma}(\mathbf{X},\mathbf{Y})) = \sum_{u\in U_{I,\sigma}} \chi_\sigma(\overline{K_u}(\mathbf{X},\mathbf{Y},\mathbf{1})).$$

By Theorem 2.19, the denominator of $\chi_\sigma(\overline{K_u}(\mathbf{X},\mathbf{Y},\mathbf{1}))$ when written in least terms is $\prod_{i=1}^{r}(1 - \mathbf{Z}^{\alpha_i})$ where $\alpha_1, \ldots, \alpha_r$ are quasigenerators of K_u and $\mathbf{Z} = (\mathbf{X},\mathbf{Y},\mathbf{1})$. Thus $\chi_\sigma(\overline{K_u}(\mathbf{Z}))|_{q\to q_\mathfrak{o}, t\to q_\mathfrak{o}^{-s}}$ has a pole at $s = 0$ if and only if there is a quasigenerator β of K_u such that $\chi_\sigma(X_1^{\beta_1} \ldots Y_{d'}^{\beta_{d+d'}})$ is a power of q^{-s}. By the definition of χ_σ in Definition 4.21, this happens only when the support of β is contained in $\{d\} \cup \{d + d' + i \mid i \in [r_\sigma]\}$.

Recall that E_σ is the monoid associated with the matrix $\Phi_\sigma \in \mathrm{Mat}_{r_\sigma\times m_\sigma}(\mathbb{Z})$ from Definition 3.8. As $w_\sigma \neq 0^{d'}1^{d'}$, there is an $h \in [2d' - 1]$ such that the h-th letter of w_σ is 1 and the $(h + 1)$-th letter of w is 0. Let i, j, k be such that $\sigma(h) = k \in [d']$ and $b^{-1}(\sigma(h + 1)) = (i, j) \in [d]^2$. Then Φ_σ has a row

$$[0^{(i-1)}, (-1)^{(j-i)}, (-2)^{(d-j+1)}, 0^{(k-1)}, 1^{(d'-k+1)}, 0, \ldots, 0, -1, 0, \ldots, 0]. \tag{5.7}$$

Note that (5.7) has the entry -1 in column d, another -1 in one other column contained in $\{d + d' + i \mid i \in [r_\sigma]\}$, and zero in the other columns contained in $\{d+d'+i \mid i \in [r_\sigma]\}$. Therefore multiplying the row (5.7) with β could not result in zero when the support of β is contained in $\{d\} \cup \{d+d' + i \mid i \in [r_\sigma]\}$. Consequently, such a tuple β could not satisfy $\Phi_\sigma\beta = 0$, and therefore not be an element of $K_u \subseteq E_\sigma$. Thus we find that $\chi_\sigma(\overline{K_u}(\mathbf{X},\mathbf{Y},\mathbf{1}))|_{q\to q_\mathfrak{o}, t\to q_\mathfrak{o}^{-s}}$ and by (5.5) also $\chi_\sigma(G_{I,\sigma}(\mathbf{X},\mathbf{Y}))|_{q\to q_\mathfrak{o}, t\to q_\mathfrak{o}^{-s}}$ cannot have a pole at $s = 0$. □

Theorem 5.13. *The rational function $\zeta_{\mathfrak{f}_{2,d}(\mathfrak{o})}(s)$ has a simple pole at $s = 0$ for all but finitely many $q_\mathfrak{o}$.*

Proof. Follows directly from Theorem 5.12 and Theorem 5.11. □

This result confirms the first part of [10, Conjecture IV ($\mathfrak{P}$-adic form)] for the relevant zeta functions (for all but possibly a finite number of $q_{\mathfrak{o}}$). (The second part is known to hold for $d \in \{2,3,4\}$, by inspection of the explicit formulas; see Corollary 7.9.)

6. Reduced and topological zeta functions

We discuss the reduced and topological zeta functions $\zeta^{\mathrm{red}}_{\mathfrak{f}_{2,d}}(t)$ and $\zeta^{\mathrm{top}}_{\mathfrak{f}_{2,d}}(s)$. Theorems 6.7 and 6.9 provide formulas for $\zeta^{\mathrm{red}}_{\mathfrak{f}_{2,d}}(t)$ and $\zeta^{\mathrm{top}}_{\mathfrak{f}_{2,d}}(s)$ respectively. Theorem 6.8 establishes that the reduced zeta function $\zeta^{\mathrm{red}}_{\mathfrak{f}_{2,d}}(t)$ has a simple pole of order $\binom{d+1}{2}$ at $t=1$. Theorems 6.10, 6.13, and 6.14 confirm parts of conjectures from [10] pertaining to the degree and pole at $s=0$ of topological subalgebra zeta functions, in the relevant special cases.

6.1. **Preliminary definitions.** In preparation, we make a few preliminary definitions. First, we define the counterpart of $\mathrm{GMC}_{I,\sigma}$ that we will use to formulate formulas for $\zeta^{\mathrm{red}}_{\mathfrak{f}_{2,d}}(t)$ and $\zeta^{\mathrm{top}}_{\mathfrak{f}_{2,d}}(s)$.

Definition 6.1. For $(I,\sigma) \in \mathcal{W}_d$, the *product of multinomial coefficients* associated with (I,σ) is $\mathrm{MC}_{I,\sigma} := \mathrm{GMC}_{I,\sigma}|_{q\to 1}$.

Second, we define integers $a_\sigma(\alpha)$ and $b_\sigma(\alpha)$ for all $\sigma \in \mathcal{S}_{2d'}$ and $\alpha \in \mathbb{N}_0^{m_\sigma}$ that are closely related to the numerical data map χ_σ.

Definition 6.2. Let $\sigma \in \mathcal{S}_{2d'}$ and $\alpha \in \mathbb{N}_0^{m_\sigma}$. Define $a_\sigma(\alpha)$ and $b_\sigma(\alpha)$ to be the respectively non-negative and positive integers such that $\chi_\sigma((\mathbf{X},\mathbf{Y},\mathbf{1})^\alpha) = (1-q^{a_\sigma(\alpha)}t^{b_\sigma(\alpha)})$.

Example 6.3. Let $\sigma = 21$, $\alpha_1 := (0,1,2,0)$, and $\alpha_2 := (0,1,0,2)$. Then

$$\chi_\sigma((\mathbf{X},\mathbf{Y},\mathbf{1})^{\alpha_1}) = \chi_\sigma(X_1^0X_2^1Y_1^21^0) = (q^1t)^0(t^2)^1(q^2t)^21^0 = q^4t^4,$$
$$\chi_\sigma((\mathbf{X},\mathbf{Y},\mathbf{1})^{\alpha_2}) = \chi_\sigma(X_1^0X_2^1Y_1^01^2) = (q^1t)^0(t^2)^1(q^2t)^01^2 = t^2.$$

Thus $a_\sigma(\alpha_1) = 4$, $b_\sigma(\alpha_1) = 4$, $a_\sigma(\alpha_2) = 0$, and $b_\sigma(\alpha_2) = 2$.

Recall the definition of $\Gamma_{I,\sigma} = \{K_u \mid u \in U_{I,\sigma}\}$ in Definition 3.23. Third, we define a subset $U_{I,\sigma,\max}$ of $U_{I,\sigma}$ and a positive rational number c_d for each $d \in \mathbb{N}_{\geqslant 2}$.

Definition 6.4. For each $(I,\sigma) \in \mathcal{W}_d$, let $U_{I,\sigma,\max}$ be the set of $u \in U_{I,\sigma}$ such that $\dim K_u = d + d' = \binom{d+1}{2}$. Let c_d be the positive rational number

$$c_d := \sum_{(I,\sigma)\in\mathcal{W}_d} \mathrm{MC}_{I,\sigma} \sum_{u\in U_{I,\sigma,\max}} \frac{|D_{\overline{K_u}}|}{\prod_{\alpha\in\mathrm{CFE}(K_u)} b_\sigma(\alpha)}, \tag{6.1}$$

where $D_{\overline{K_u}}$ is defined in (2.6).

Remark 6.5. Although it is not a priori clear, c_d does not depend on the family $\Gamma_{I,\sigma} = \{K_u \mid u \in U_{I,\sigma}\}$ of simplicial monoids, but only on d. This is a consequence of Theorem 6.8.

We cannot offer a conceptual interpretation of c_d. Its values for $d \leqslant 6$ are tabulated in Table 6.1.

d	2	3	4	5	6
c_d	$\frac{3}{4}$	$\frac{25}{54}$	$\frac{569}{2304}$	$\frac{3800243}{32400000}$	$\frac{8743819}{172800000}$

Table 6.1. Values of c_d for each $d \in \{2,3,4,5,6\}$.

6.2. **Reduced zeta functions.** The reduced zeta function $\zeta^{\mathrm{red}}_{\mathfrak{f}_{2,d}}(t)$ can be obtained by substituting $q \to 1$ in $\zeta_{\mathfrak{f}_{2,d}}(q,t)$. We straightforwardly adapt Theorem 4.24 to a formula for $\zeta^{\mathrm{red}}_{\mathfrak{f}_{2,d}}(t)$ and then use it to determine the order and residue of the pole at $t = 1$ of $\zeta^{\mathrm{red}}_{\mathfrak{f}_{2,d}}(t)$. We first define a reduced counterpart of the numerical data map.

Definition 6.6. The *reduced numerical data map* χ_{red} is

$$\chi_{\mathrm{red}} : \mathbb{Q}(\mathbf{X},\mathbf{Y}) \to \mathbb{Q}(t) : X_i \mapsto t^i, Y_j \mapsto t^j.$$

The following theorem is a straightforward adaption of Theorem 4.24.

Theorem 6.7. *For all $d \in \mathbb{N}_{\geqslant 2}$,*

$$\zeta^{\mathrm{red}}_{\mathfrak{f}_{2,d}}(t) = \sum_{(I,\sigma)\in\mathcal{W}_d} \mathrm{MC}_{I,\sigma}\, \chi_{\mathrm{red}}(G_{I,\sigma}(\mathbf{X},\mathbf{Y})).$$

Proof. Follows from Theorem 4.24 after substituting $q \to 1$ on both sides of (4.5). ☐

Next, we will use this formula to deduce the order and residue of the pole at $t = 1$ of $\zeta^{\mathrm{red}}_{\mathfrak{f}_{2,d}}(t)$. Recall that $D = d + d' = \binom{d+1}{2}$ is the $\mathbb{Z}$-rank of $\mathfrak{f}_{2,d}$.

Theorem 6.8. *The reduced zeta function $\zeta^{\mathrm{red}}_{\mathfrak{f}_{2,d}}(t)$ has a pole at $t = 1$ of order D with residue*

$$\lim_{t\to 1}\left((t-1)^D \zeta^{\mathrm{red}}_{\mathfrak{f}_{2,d}}(t)\right) = (-1)^D c_d,$$

where c_d is defined in Definition 6.4.

Proof. Recall from Proposition 3.22 that $G_{I,\sigma}$ can be seen as the projection of the subset $I_{E_\sigma,A_{I,\sigma},C_{I,\sigma}} \in \mathbb{N}_0^{m_\sigma}$ on the first $d+d'$ coordinates, or equivalently $G_{I,\sigma}(\mathbf{X},\mathbf{Y}) = I_{E_\sigma,A_{I,\sigma},C_{I,\sigma}}(\mathbf{X},\mathbf{Y},\mathbf{1})$. Let $\Gamma_{I,\sigma} = \{K_u \mid u \in U_{I,\sigma}\}$ be as in Definition 3.23. Using that $\bigcup_{u\in U_{I,\sigma}} \overline{K_u} = I_{E_\sigma,A_{I,\sigma},C_{I,\sigma}}$ in Theorem 6.7 results in

$$\zeta^{\mathrm{red}}_{\mathfrak{f}_{2,d}}(t) = \sum_{(I,\sigma)\in\mathcal{W}_d} \mathrm{MC}_{I,\sigma} \sum_{u\in U_{I,\sigma}} \chi_{\mathrm{red}}(\overline{K_u}(\mathbf{X},\mathbf{Y},\mathbf{1})). \tag{6.2}$$

By Theorem 2.19,

$$\overline{K_u}(\mathbf{Z}) = \frac{\sum_{\beta\in D_{\overline{K_u}}} \mathbf{Z}^\beta}{\prod_{\alpha\in\mathrm{CFE}(K_u)}(1-\mathbf{Z}^\alpha)},$$

where $D_{\overline{K_u}}$ is defined in (2.6) and $\mathbf{Z} = (\mathbf{X},\mathbf{Y},\mathbf{1})$. Applying χ_{red} on both sides results in

$$\chi_{\mathrm{red}}\left(\overline{K_u}(\mathbf{Z})\right) = \frac{\sum_{\beta\in D_{\overline{K_u}}} \chi_{\mathrm{red}}(\mathbf{Z}^\beta)}{\prod_{\alpha\in\mathrm{CFE}(K_u)}(1-t^{b_\sigma(\alpha)})},$$

where $b_\sigma(\alpha)$ was defined in Definition 6.2. It follows that $\chi_{\mathrm{red}}(\overline{K_u}(\mathbf{X},\mathbf{Y},\mathbf{1}))$ has a pole at $t = 1$ of order $\dim K_u$ and the residue of this pole is

$$\lim_{t\to 1}\left((t-1)^{\dim K_u}\overline{K_u}(\mathbf{Z})\right) = (-1)^{\dim K_u} \frac{|D_{\overline{K_u}}|}{\prod_{\alpha\in\mathrm{CFE}(K_u)} b_\sigma(\alpha)}. \tag{6.3}$$

Recall from Remark 3.16 that the dimension of $G_{I,\sigma}$, and therefore also of $I_{E_\sigma,A_{I,\sigma},C_{I,\sigma}}$ and K_u, is at most D. Thus the summand indexed by $u \in U_{I,\sigma}$ in (6.2) has a pole at $t = 1$ of order D if $\dim \mathcal{K}_u = D$ and otherwise the pole has a lower order. Therefore $\zeta^{\mathrm{red}}_{\mathfrak{f}_{2,d}}(t)$ has a pole at $t = 1$ of order at most D. To prove that the order is exactly D, it suffices to show that the sum of the residues of the summands in (6.2) with maximal pole order does not vanish. In other words, we need to show that summing (6.3) over all $u \in U_{I,\sigma}$ with $\dim K_u = D$ does not cancel out. This is of course trivial, because (6.3) has the same sign for all these u. To find the residue of $\zeta^{\mathrm{red}}_{\mathfrak{f}_{2,d}}(t)$, we may just sum (6.3) over all these u, resulting in $(-1)^D c_d$, where c_d was defined in Definition 6.4. ☐

6.3. **Topological zeta functions.** Next, we study the topological zeta function $\zeta^{\mathrm{top}}_{\mathfrak{f}_{2,d}}(s)$. We write down a formula in Theorem 6.9 and determine its degree in Theorem 6.10. In Theorem 6.11, we link its behaviour at infinity to the behaviour at $t = 1$ of $\zeta^{\mathrm{red}}_{\mathfrak{f}_{2,d}}(t)$. In Section 6.4 we determine the behaviour of $\zeta^{\mathrm{top}}_{\mathfrak{f}_{2,d}}(s)$ at $s = 0$.

In Section 1.2, we informally introduced the topological zeta function $\zeta^{\mathrm{top}}_{\mathfrak{f}_{2,d}}(s)$ as the rational function in s obtained as the first non-zero coefficient of the $\mathfrak{p}$-adic zeta function $\zeta_{\mathfrak{f}_{2,d}}(q, q^{-s})$, expanded in $q - 1$. More precisely,

$$\zeta^{\mathrm{top}}_{\mathfrak{f}_{2,d}}(s) := \lim_{q\to 1}(q-1)^{\mathrm{rank}\ \mathfrak{f}_{2,d}}\zeta_{\mathfrak{f}_{2,d}}(q, q^{-s}).$$

For example, for $a \in \mathbb{N}_0$ and $b \in \mathbb{N}$,

$$\lim_{q\to 1}(q-1)\frac{1}{1-q^{a-bs}} = \frac{1}{bs-a}. \tag{6.4}$$

The following theorem is an adaption of Theorem 4.24 to a formula for $\zeta^{\mathrm{top}}_{\mathfrak{f}_{2,d}}(s)$.

Theorem 6.9. *For $d \in \mathbb{N}_{\geqslant 2}$,*

$$\zeta^{\mathrm{top}}_{\mathfrak{f}_{2,d}}(s) = \sum_{(I,\sigma)\in\mathcal{W}_d} \mathrm{MC}_{I,\sigma} \sum_{u\in U_{I,\sigma,\max}} \frac{|D_{\overline{K_u}}|}{\prod_{\alpha\in\mathrm{CFE}(K_u)}(b_\sigma(\alpha)s - a_\sigma(\alpha))}.$$

Proof. Recall from Proposition 3.22 that $G_{I,\sigma}$ can be seen as the projection of the subset $I_{E_\sigma,A_{I,\sigma},C_{I,\sigma}} \subseteq \mathbb{N}_0^{m_\sigma}$ on the first $d+d'$ coordinates, or equivalently $G_{I,\sigma}(\mathbf{X},\mathbf{Y}) = I_{E_\sigma,A_{I,\sigma},C_{I,\sigma}}(\mathbf{X},\mathbf{Y},\mathbf{1})$. Let $\Gamma_{I,\sigma} = \{K_u \mid u \in U_{I,\sigma}\}$ be as in Definition 3.23. Using that $\bigcup_{u\in U_{I,\sigma}} \overline{K_u} = I_{E_\sigma,A_{I,\sigma},C_{I,\sigma}}$ in Theorem 4.24 results in

$$\zeta_{\mathfrak{f}_{2,d}}(q,t) = \sum_{(I,\sigma)\in\mathcal{W}_d} \mathrm{GMC}_{I,\sigma} \sum_{u\in U_{I,\sigma}} \chi_\sigma(\overline{K_u}(\mathbf{X},\mathbf{Y},\mathbf{1})).$$

By Theorem 2.19,

$$\chi_\sigma(\overline{K_u}(\mathbf{Z})) = \frac{\sum_{\beta\in D_{\overline{K_u}}}\chi_\sigma(\mathbf{Z}^\beta)}{\prod_{\alpha\in\mathrm{CFE}(K_u)}(1-\chi_\sigma(\mathbf{Z}^\alpha))}, \tag{6.5}$$

where $\mathbf{Z} = (\mathbf{X},\mathbf{Y},\mathbf{1})$. Note that $\lim_{q\to 1}(q-1)^{d+d'}\chi_\sigma(\overline{K_u}(\mathbf{Z}))$ vanishes if $\dim K_u < d+d'$, that is, $u \in U_{I,\sigma}\backslash U_{I,\sigma,\max}$. The numerator of (6.5) becomes $|D_{\overline{K_u}}|$ after substituting $q \to 1$. For each $u \in U_{I,\sigma,\max}$, the denominator of (6.5) becomes $\prod_{\alpha\in\mathrm{CFE}(K_u)}(b_\sigma(\alpha)s - a_\sigma(\alpha))$ after multiplication with $(q-1)^{-d-d'}$ and taking the limit $q \to 1$, cf. (6.4) and Definition 6.2. □

The degree of a rational expression is the degree of the numerator minus the degree of the denominator. The following confirms [10, Conj. I] for the considered algebras.

Theorem 6.10. *The topological zeta function $\zeta^{\mathrm{top}}_{\mathfrak{f}_{2,d}}(s)$ has degree $-D$ in s:*

$$\deg_s\left(\zeta^{\mathrm{top}}_{\mathfrak{f}_{2,d}}(s)\right) = -D,$$

where $D = d + d' = \binom{d+1}{2}$ is the $\mathbb{Z}$-rank of $\mathfrak{f}_{2,d}$.

Proof. All summands in (6.9) have degree $-D$ in s. For each $u \in U_{I,\sigma,\max}$, the numerator of the summand corresponding to u is $\mathrm{MC}_{I,\sigma}|D_{\mathcal{K}_u}|$, which is always positive. Similarly, the highest degree coefficient of the denominator of the summand corresponding to u is $\prod_{\alpha\in\mathrm{CFE}(K_u)} b_\sigma(\alpha)$, which is also always positive. Therefore cancellation of the highest degree terms in (6.9) is not possible. □

Next, we study the behaviour of the topological zeta function at infinity.

Theorem 6.11. *The topological zeta function $\zeta^{\mathrm{top}}_{\mathfrak{f}_{2,d}}(s)$ satisfies*

$$\lim_{s\to 0} s^{-D}\zeta^{\mathrm{top}}_{\mathfrak{f}_{2,d}}(s^{-1}) = c_d,$$

where $D = d + d' = \binom{d+1}{2}$ is the $\mathbb{Z}$-rank of $\mathfrak{f}_{2,d}$, and c_d is defined in Definition 6.4.

Proof. Substituting s^{-1} for s in the summand corresponding to $u \in U_{I,\sigma,\max}$ in (6.9), multiplying by s^{-D} and taking the limit $s \to 0$ results in

$$\lim_{s\to 0} s^{-D}\frac{|D_{\overline{K_u}}|}{\prod_{\alpha\in\mathrm{CFE}(K_u)}(b_\sigma(\alpha)s^{-1} - a_\sigma(\alpha))} = \lim_{s\to 0}\frac{|D_{\overline{K_u}}|}{\prod_{\alpha\in\mathrm{CFE}(K_u)}(b_\sigma(\alpha) - a_\sigma(\alpha)s)} = \frac{|D_{\overline{K_u}}|}{\prod_{\alpha\in\mathrm{CFE}(K_u)} b_\sigma(\alpha)}.$$

Therefore, using (6.9), we find

$$\lim_{s\to 0} s^{-D}\zeta^{\mathrm{top}}_{\mathfrak{f}_{2,d}}(s^{-1}) = \sum_{(I,\sigma)\in\mathcal{W}_d}\mathrm{MC}_{I,\sigma}\sum_{u\in U_{I,\sigma,\max}}\frac{|D_{\overline{K_u}}|}{\prod_{\alpha\in\mathrm{CFE}(K_u)} b_\sigma(\alpha)},$$

which is the definition of c_d in (6.1). □

The following corollary shows that the behaviour at infinity of $\zeta^{\mathrm{top}}_{\mathfrak{f}_{2,d}}(s)$ is closely related to the behaviour at $t = 1$ of $\zeta^{\mathrm{red}}_{\mathfrak{f}_{2,d}}(t)$.

Corollary 6.12. *For $d \in \mathbb{N}_{\geqslant 2}$,*

$$\lim_{s\to 0} s^{-D}\zeta^{\mathrm{top}}_{\mathfrak{f}_{2,d}}(s^{-1}) = \lim_{t\to 1}(1-t)^D\zeta^{\mathrm{red}}_{\mathfrak{f}_{2,d}}(t),$$

where $D = d + d' = \binom{d+1}{2}$ is the $\mathbb{Z}$-rank of $\mathfrak{f}_{2,d}$.

Proof. Combine Theorem 6.11 with Theorem 6.8. □

Corollary 6.12 may be compared with [9, Conjecture 6.7], which describes an analogous phenomenon for topological and reduced zeta functions associated with graded ideal zeta functions of free nilpotent Lie rings of arbitrary rank and nilpotency class.

6.4. **Behaviour at zero of the topological zeta function.** In Theorem 6.13, we show that the topological zeta function $\zeta^{\mathrm{top}}_{\mathfrak{f}_{2,d}}(s)$ has a simple pole at $s = 0$, just as the $\mathfrak{p}$-adic zeta function $\zeta_{\mathfrak{f}_{2,d}(\mathfrak{o})}(s)$ for all but possibly a finite number of $q_{\mathfrak{o}}$ (see Theorem 5.13). The residue of this pole is determined in Theorem 6.14. These two theorems together confirm [10, Conj. IV (Topological form)] for the considered subalgebra zeta functions.

Theorem 6.13. *The topological zeta function $\zeta^{\mathrm{top}}_{\mathfrak{f}_{2,d}}(s)$ has a simple pole at $s = 0$.*

Proof. We start from the formula for $\zeta^{\mathrm{top}}_{\mathfrak{f}_{2,d}}(s)$ in Theorem 6.9. Let $(I,\sigma) \in \mathcal{W}_d$ and $u \in U_{I,\sigma,\max}$. The summand corresponding to u in (6.9) has numerator $\mathrm{MC}_{I,\sigma}|D_{\overline{K_u}}|$ and denominator $\prod_{\alpha\in\mathrm{CFE}(K_u)}(b_\sigma(\alpha)s - a_\sigma(\alpha))$. This numerator is a positive rational number and the denominator is zero at $s = 0$ if and only if there is a $\alpha \in \mathrm{CFE}(K_u)$ such that $a_\sigma(\alpha)$ is zero. Recall from Definition 6.2 that $a_\sigma(\alpha)$ is the non-negative integer such that $\chi_\sigma((\mathbf{X},\mathbf{Y},\mathbf{1})^\alpha) = (1 - q^{a_\sigma(\alpha)}t^{b_\sigma(\alpha)})$. Looking at Definition 4.21, $a_\sigma(\alpha)$ is zero if and only if the support of α is contained in $\{d\} \cup \{d + d' + i \mid i \in [r_\sigma]\}$. Thus the summand corresponding to u in (6.9) has a pole at $s = 0$ if and only if there is a completely fundamental element of K_u with support contained in $\{d\} \cup \{d+d'+i \mid i \in [r_\sigma]\}$. From the proof of Theorem 5.12, we know that this cannot happen if w_σ is not the trivial Dyck word. From the proof of Theorem 5.11, we know that if w_σ

is the trivial Dyck word, then there are K_u that have such a completely fundamental element, and, moreover, no K_u can have more than one such completely fundamental element. Thus the summand corresponding to u in (6.9) has at most a simple pole at $s = 0$, and therefore $\zeta^{\mathrm{top}}_{\mathfrak{f}_{2,d}}(s)$ has at most a simple pole.

Let

$$U_{0,\max} := \{u \in U_{I,\sigma,\max} \mid (I, \sigma) \in \mathcal{W}_d, \delta_d + 2\delta_{d+d'+1} \in K_u\}.$$

The summand corresponding to u in (6.9) has a simple pole at $s = 0$ if and only if $u \in U_{0,\max}$. By the reasoning earlier in this proof, if $u \in U_{0,\max}$ then $u \in U_{I,\sigma,\max}$ with $w_\sigma = 0^{d'}1^{d'}$. Also $I = [d-1]$ and $J_\sigma = [d'-1]$, because otherwise $U_{I,\sigma,\max}$ is empty. It follows using Lemma 5.4 that if $u \in U_{0,\max} \cap U_{I,\sigma}$, then

$$\mathrm{MC}_{I,\sigma} = \binom{d}{[d-1]}\binom{d'}{[d'-1]} = d!d'!.$$

The residue of $\zeta^{\mathrm{top}}_{\mathfrak{f}_{2,d}}(s)$ at $s = 0$ is therefore

(6.6)
$$\lim_{s\to 0} s\zeta^{\mathrm{top}}_{\mathfrak{f}_{2,d}}(s) = d!d'! \sum_{u \in U_{0,\max}} \frac{|D_{\overline{K_u}}|}{b_\sigma(\delta_d + 2\delta_{d+d'+1}) \prod_{\alpha \in \mathrm{CFE}(K_u)\setminus\{\delta_d + 2\delta_{d+d'+1}\}}(-a_\sigma(\alpha))}.$$

These summands all have the same sign $(-1)^{d+d'-1}$, therefore there is no cancellation and $\zeta^{\mathrm{top}}_{\mathfrak{f}_{2,d}}(s)$ indeed has a simple pole at $s = 0$. □

Next, we simplify the complicated expression for the residue in (6.6).

Theorem 6.14. *The residue of the simple pole at $s = 0$ of the topological zeta function $\zeta^{\mathrm{top}}_{\mathfrak{f}_{2,d}}(s)$ is*

$$\lim_{s\to 0} s\zeta^{\mathrm{top}}_{\mathfrak{f}_{2,d}}(s) = \frac{(-1)^{D-1}}{(D-1)!},$$

where $D = d + d' = \binom{d+1}{2}$ is the $\mathbb{Z}$-rank of $\mathfrak{f}_{2,d}$.

Proof. By the proof of Theorem 6.13, we know that the summands of (6.9) indexed by $(I, \sigma) \in \mathcal{W}_d$ with $w_\sigma \neq 0^{d'}1^{d'}$ do not contribute to the residue. Therefore the residue can be written as

$$\lim_{s\to 0} s\zeta^{\mathrm{top}}_{\mathfrak{f}_{2,d}}(s) = \lim_{s\to 0} s \sum_{\substack{(I,\sigma)\in\mathcal{W}_d \\ w_\sigma = 0^{d'}1^{d'}}} \mathrm{MC}_{I,\sigma} \sum_{u \in U_{I,\sigma,\max}} \frac{|D_{\overline{K_u}}|}{\prod_{\alpha \in \mathrm{CFE}(K_u)}(b_\sigma(\alpha)s - a_\sigma(\alpha))}.$$

The proof of Theorem 6.13 also showed that the nonzero summands all have $\mathrm{MC}_{I,\sigma} = d!d'!$, which is independent of (I, σ), thus

$$\lim_{s\to 0} s\zeta^{\mathrm{top}}_{\mathfrak{f}_{2,d}}(s) = d!d'! \lim_{s\to 0} s \sum_{\substack{(I,\sigma)\in\mathcal{W}_d \\ w_\sigma = 0^{d'}1^{d'}}} \sum_{u \in U_{I,\sigma,\max}} \frac{|D_{\overline{K_u}}|}{\prod_{\alpha \in \mathrm{CFE}(K_u)}(b_\sigma(\alpha)s - a_\sigma(\alpha))}.$$

Recall that

$$\frac{|D_{\overline{K_u}}|}{\prod_{\alpha \in \mathrm{CFE}(K_u)}(b_\sigma(\alpha)s - a_\sigma(\alpha))} = \lim_{q\to 1}(q-1)^{d+d'}\chi_\sigma(\overline{K_u}(\mathbf{X}, \mathbf{Y}, 1))|_{t\to q^{-s}}$$

and $\bigcup_{u \in U_{I,\sigma}} \overline{K_u} = I_{E_\sigma, A_{I,\sigma}, C_{I,\sigma}}$ where the union is disjoint. This allows us to write the residue as

$$\lim_{s\to 0} s\zeta^{\mathrm{top}}_{\mathfrak{f}_{2,d}}(s) = d!d'! \lim_{s\to 0} s \lim_{q\to 1}(q-1)^{d+d'} \sum_{\substack{(I,\sigma)\in\mathcal{W}_d \\ w_\sigma = 0^{d'}1^{d'}}} \chi_{\mathrm{n.o.}}(I_{E_{\mathrm{n.o.}}, A_{I,J_\sigma}, C_{I,J_\sigma}}(\mathbf{X}, \mathbf{Y}, 1))|_{t\to q^{-s}}.$$

$\alpha \in \mathrm{CFE}(E_0)$	$\chi_{\text{n.o.}}((\mathbf{X},\mathbf{Y},1)^{\alpha})$	$a_{\text{n.o.}}(\alpha)$	$b_{\text{n.o.}}(\alpha)$
δ_i with $i \in [d-2]$	$(1-q^{i(d-i)}t^i)$	$i(d-i)$	i
$\delta_{d-1}+\delta_{d+d'+1}$	$(1-q^{d-1}t^{d-1})$	$d-1$	$d-1$
$\delta_d + 2\delta_{d+k}$ with $k \in [d']$	$(1-q^{2k(d+d'-k)}t^{d+2k})$	$2k(d+d'-k)$	$d+2k$
$\delta_d + 2\delta_{d+d'+1}$	$(1-t^d)$	0	d

TABLE 6.2. The completely fundamental elements of E_0 and the corresponding $a_{\text{n.o.}}(\alpha)$ and $b_{\text{n.o.}}(\alpha)$.

Recall from Remark 3.35 that $\bigcup_{(I,\sigma)\in \mathcal{W}_d, w_\sigma = 0^{d'}1^{d'}} I_{E_{\text{n.o.}},A_{I,J_\sigma},C_{I,J_\sigma}} = E_{\text{n.o.}}$ and this union is disjoint. Therefore the residue can be written as

$$\lim_{s\to 0} s\zeta^{\text{top}}_{\mathfrak{f}_{2,d}}(s) = d!d'! \lim_{s\to 0} s \lim_{q\to 1}(q-1)^{d+d'}\chi_{\text{n.o.}}(E_{\text{n.o.}}(\mathbf{X},\mathbf{Y},1))|_{t\to q^{-s}}.$$

Now recall the subset E_0 of $E_{\text{n.o.}}$ from Definition 3.26 and consider the triangulation $\Gamma = \{K_u \mid u \in U_{\text{n.o.}}\}$ of $E_{\text{n.o.}}$ in Proposition 3.28. Let $U_0 := \{u \in U_{\text{n.o.}} \mid K_u \subseteq E_0\}$ and $U_0^c := \{u \in U_{\text{n.o.}} \mid K_u \nsubseteq E_0\}$. Then $E_0 = \bigcup_{u\in U_0} \overline{K}_u$ and this union is disjoint. Thus the residue becomes

$$\lim_{s\to 0} s\zeta^{\text{top}}_{\mathfrak{f}_{2,d}}(s) = d!d'! \lim_{s\to 0} s \lim_{q\to 1}(q-1)^{d+d'}\chi_{\text{n.o.}}\left(E_0(\mathbf{X},\mathbf{Y},1) + \sum_{u\in U_0^c}\overline{K_u}(\mathbf{X},\mathbf{Y},1)\right)|_{t\to q^{-s}}.$$

The triangulation $\Gamma = \{K_u \mid u \in U_{\text{n.o.}}\}$ was constructed so that the K_u for $u \in U_0^c$ do not contain $\delta_d + 2\delta_{d+d'+1}$. Therefore by the same reasoning as in the proof of Theorem 6.13, the summands $\chi_{\text{n.o.}}(\overline{K_u}(\mathbf{X},\mathbf{Y},1))$ do not contribute to the residue. For every $\alpha \in \mathrm{CFE}(E_0)$, let $a_{\text{n.o.}}(\alpha)$ and $b_{\text{n.o.}}(\alpha)$ be the respectively non-negative and positive integers such that $\chi_{\text{n.o.}}((\mathbf{X},\mathbf{Y},1)^{\alpha_i}) = (1-q^{a_{\text{n.o.}}(\alpha)}t^{b_{\text{n.o.}}(\alpha)})$. As E_0 is simplicial (see Remark 3.27) we may use Theorem 2.19 to deduce

$$\lim_{q\to 1}(q-1)^{d+d'}\chi_{\text{n.o.}}(E_0(\mathbf{X},\mathbf{Y},1)|_{t\to q^{-s}} = \frac{|D_{E_0}|}{\prod_{\alpha\in\mathrm{CFE}(E_0)}(b_{\text{n.o.}}(\alpha)s - a_{\text{n.o.}}(\alpha))}.$$

Thus the residue becomes

$$\begin{aligned}\lim_{s\to 0} s\zeta^{\text{top}}_{\mathfrak{f}_{2,d}}(s) &= d!d'! \lim_{s\to 0} s \frac{|D_{E_0}|}{\prod_{\alpha\in\mathrm{CFE}(E_0)}(b_{\text{n.o.}}(\alpha)s - a_{\text{n.o.}}(\alpha))} \\ &= (-1)^{D-1}d!d'! \frac{|D_{E_0}|}{b_{\text{n.o.}}(\delta_d + 2\delta_{d+d'+1})\prod_{\alpha\in\mathrm{CFE}(E_0)\setminus\{\delta_d+2\delta_{d+d'+1}\}} a_{\text{n.o.}}(\alpha)}. \qquad (6.7)\end{aligned}$$

The completely fundamental elements α of E_0 and the corresponding data $a_{\text{n.o.}}(\alpha)$ and $b_{\text{n.o.}}(\alpha)$ are listed in Table 6.2. Using this data, the denominator in (6.7) becomes

$$d\left(\prod_{i\in[d-2]} i(d-i)\right)(d-1)\left(\prod_{k\in[d']} 2k(d+d'-k)\right),$$

which simplifies to $2^{d'}(d)!(d')!(d+d'-1)!$.

Lastly, we determine $|D_{E_0}|$. To do this, we need to count the number of elements $x \in E_0$ that can be written as a $\mathbb{Q}$-linear combination of the completely fundamental elements of E_0 with coefficients in $[0,1)$. Since δ_i is the only completely fundamental element of E_0 with support containing $\{i\}$ for $i \in [d-2]$, we deduce that the coefficient of δ_i needs to be zero for $i \in [d-2]$. Similarly, $\delta_{d-1} + \delta_{d+d'+1}$ is the only completely fundamental element with support containing $\{d-1\}$, thus the coefficient of $\delta_{d-1} + \delta_{d+d'+1}$ is zero as well. Also, $\delta_d + 2\delta_{i+1}$ is the only completely fundamental element with support containing $\{i+1\}$ for $i \in d-1+[d']$, thus its coefficient lies in $\{0, 1/2\}$.

Since the coefficient of $\delta_{d-1} + \delta_{d+d'+1}$ is zero, $\delta_d + 2\delta_{d+d'+1}$ is the only remaining completely fundamental element with support containing $\{d + d' + 1\}$. Therefore its coefficient lies in $\{0, 1/2\}$ as well. Thus we find that

$$D_{E_0} = E_0 \cap \left\{ \sum_{i\in d-1+[d'+1]} a_i(\delta_d + 2\delta_{i+1}) \middle| a_i \in \{0, 1/2\} \right\}.$$

Obviously $\sum_{i\in d-1+[d'+1]} a_i(\delta_d+2\delta_{i+1})$ lies in $\mathbb{N}_0^{d+d'+1}$ if and only if $\sum_{i\in d-1+[d'+1]} a_i \in \mathbb{N}$, in other words when an even number of a_i are non-zero. Moreover, in that case, it also lies in E_0, thus we find that $|D_{E_0}| = 2^{d'}$.

We conclude by inputting this data in (6.7):

$$\lim_{s\to 0} s\zeta^{\mathrm{top}}_{\mathfrak{f}_{2,d}}(s) = (-1)^D d!d'! \frac{2^{d'}}{2^{d'}(d)!(d')!(d + d' - 1)!} = \frac{(-1)^D}{(D - 1)!}. \qquad \square$$

7. Explicit computations

We record (aspects of) explicit computations of the $\mathfrak{p}$-adic, reduced, and topological zeta functions. The full results are available at 10.5281/zenodo.7966735. We start by collecting the well-known formulas for $d = 2, 3$.

Proposition 7.1 ($d = 2$)**.**

$$\zeta_{\mathfrak{f}_{2,2}}(q, t) = \frac{1 - q^3t^3}{(1 - q^3t^2)(1 - q^2t^2)(1 - t)(1 - qt)},$$

$$\zeta^{\mathrm{top}}_{\mathfrak{f}_{2,2}}(s) = \frac{3}{2(2s - 3)(s - 1)s},$$

$$\zeta^{\mathrm{red}}_{\mathfrak{f}_{2,2}}(t) = \frac{t^2 + t + 1}{(1 - t^2)^2(1 - t)}.$$

Proof. The $\mathfrak{p}$-adic formula was given in [8, Prop. 8.1], the others follow immediately. $\square$

Proposition 7.2 ($d = 3$)**.**

$$\zeta_{\mathfrak{f}_{2,3}}(q, t) = \frac{(1 - q^8t^4)W_{2,3}(q, t)}{(1 - t)(1 - qt)(1 - q^2t)(1 - q^4t^2)(1 - q^5t^2)(1 - q^6t^2)(1 - q^6t^3)(1 - q^7t^3)},$$

where $W_{2,3}(X, Y)$ *is*

$$1 + X^3Y^2 + X^4Y^2 + X^5Y^2 - X^4Y^3 - X^5Y^3 - X^6Y^3 - X^7Y^4 - X^9Y^4$$
$$- X^{10}Y^5 - X^{11}Y^5 - X^{12}Y^5 + X^{11}Y^6 + X^{12}Y^6 + X^{13}Y^6 + X^{16}Y^8.$$

Furthermore

$$\zeta^{\mathrm{top}}_{\mathfrak{f}_{2,3}}(s) = \frac{25s^2 - 94s + 84}{3(3s - 7)(3s - 8)(2s - 5)(s - 1)(s - 2)^2(s - 3)s},$$

$$\zeta^{\mathrm{red}}_{\mathfrak{f}_{2,3}}(t) = \frac{t^8 + 2t^7 + 7t^6 + 9t^5 + 12t^4 + 9t^3 + 7t^2 + 2t + 1}{(1 - t^3)^3(1 - t^2)^2(1 - t)}.$$

Proof. The $\mathfrak{p}$-adic formula was given in [15, Thm 24], the others follow immediately. $\square$

In [2], an algorithm is presented to write the generating function enumerating integral points of a convex pointed polyhedral cone in a closed form. This algorithm is implemented in the software package `LattE` [1], which can be accessed in `SageMath` [12] through the package `Zeta` [11]. By this route, we were able to implement Theorem 4.24 and recover the explicit expressions for $\zeta_{\mathfrak{f}_{2,d}}(q, t)$ for $n = 2, 3$ in Propositions 7.1 and

7.2. Moreover, we were also able to obtain an explicit expression for $\zeta_{\mathfrak{f}_{2,4}}(q,t)$, which was not known before.

Theorem 7.3 ($d = 4$, $\mathfrak{p}$-adic)**.** *There is an explicitly determined polynomial* $\Psi_{2,4}(X, Y) \in \mathbb{Z}[X,Y]$ *of degrees* 335 *in* X *and* 88 *in* Y *such that*

$$\zeta_{\mathfrak{f}_{2,4}(\mathfrak{o})}(q,t) = \frac{\Phi_{2,4}(q,t)}{\Psi_{2,4}(q,t)}, \tag{7.1}$$

where $\Psi_{2,4}(q,t)$ *is*

$$\begin{aligned}&(1-q^{27}t^7)(1-q^{25}t^7)(1-q^{25}t^6)(1-q^{28}t^7)(1-q^{22}t^5)^2(1-q^{21}t^5)(1-q^{17}t^4)\\&(1-q^{15}t^4)(1-q^{13}t^4)(1-q^{26}t^6)(1-q^{13}t^3)(1-q^{11}t^3)(1-q^{18}t^4)(1-q^9t^2)\\&(1-q^{12}t^3)(1-q^{24}t^6)(1-q^{16}t^4)(1-q^{14}t^4)(1-q^9t^3)(1-q^{12}t^4)(1-qt)(1-t).\end{aligned}$$

Corollary 7.4 ($d = 4$, reduced)**.**

$$\zeta^{\mathrm{red}}_{\mathfrak{f}_{2,4}}(t) = \frac{\Phi^{\mathrm{red}}_{2,4}(t)}{(1-t)^2(1-t^3)^4(1-t^4)^4},$$

where $\Phi^{\mathrm{red}}_{2,4}(t)$ *is*

$$\begin{aligned}&t^{20}+2t^{19}+15t^{18}+30t^{17}+87t^{16}+156t^{15}+284t^{14}+414t^{13}+562t^{12}+658t^{11}\\&+703t^{10}+658t^9+562t^8+414t^7+284t^6+156t^5+87t^4+30t^3+15t^2+2t+1.\end{aligned}$$

Proof. Substitute $q = 1$ in (7.1). Alternatively, explicate [7, Prop. 4.1]. ☐

Theorem 7.5 ($d = 4$, topological)**.**

$$\begin{aligned}(\Phi^{\mathrm{top}}_{2,4}(s)/(168\,\zeta^{\mathrm{top}}_{\mathfrak{f}_{2,4}}(s))) =&(7s-25)(7s-27)(6s-25)(5s-21)(5s-22)^2\\&(4s-13)(4s-15)(4s-17)(3s-11)(3s-13)^2\\&(2s-7)(2s-9)^2(s-1)(s-3)^2(s-4)^4s,\end{aligned}$$

where $\Phi^{\mathrm{top}}_{2,4}(s)$ *is*

$$\begin{aligned}&21078036000s^{13}-1040066363064s^{12}+23656166485364s^{11}\\&-328379597912246s^{10}+3103756047141233s^9-21092307321737791s^8\\&+106022910302150804s^7-399106101276334990s^6+1125038325014124489s^5\\&-2345400850582061927s^4+3514612915281294714s^3\\&-3584726815997417886s^2+2230351512292203300s-639268261271640000.\end{aligned}$$

The computation of the $\mathfrak{p}$-adic zeta function for $d = 5$ is currently out of our reach. We are, however, able to compute the reduced zeta function $\zeta^{\mathrm{red}}_{\mathfrak{f}_{2,5}}(s)$ (using Evseev's method, [7, Prop. 4.1]) and the topological zeta function $\zeta^{\mathrm{top}}_{\mathfrak{f}_{2,5}}(s)$ (using our method).

Theorem 7.6 ($d = 5$, reduced; [7])**.**

$$\zeta^{\mathrm{red}}_{\mathfrak{f}_{2,5}}(t) = \frac{\Phi^{\mathrm{red}}_{2,5}(t)}{((1-t^5)^5(1-t^3)^5(1-t^4)^4(1-t))},$$

where $\Phi^{\mathrm{red}}_{2,5}(t)$ *is*

$$\begin{aligned}
&t^{42} + 4t^{41} + 30t^{40} + 115t^{39} + 431t^{38} + 1330t^{37} + 3709t^{36} + 9185t^{35} + 20876t^{34}\\
&+ 43410t^{33} + 83737t^{32} + 150127t^{31} + 252056t^{30} + 397040t^{29} + 589457t^{28}\\
&+ 826057t^{27} + 1095916t^{26} + 1377780t^{25} + 1644507t^{24} + 1864452t^{23} + 2010117t^{22}\\
&+ 2060784t^{21} + 2010117t^{20} + 1864452t^{19} + 1644507t^{18} + 1377780t^{17} + 1095916t^{16}\\
&+ 826057t^{15} + 589457t^{14} + 397040t^{13} + 252056t^{12} + 150127t^{11} + 83737t^{10}\\
&+ 43410t^{9} + 20876t^{8} + 9185t^{7} + 3709t^{6} + 1330t^{5} + 431t^{4} + 115t^{3} + 30t^{2} + 4t + 1.
\end{aligned}$$

Theorem 7.7 ($d = 5$, topological)**.**

$$\zeta^{\mathrm{top}}_{\mathfrak{f}_{2,5}}(s) = \frac{\Phi^{\mathrm{top}}_{2,5}(s)}{\Psi^{\mathrm{top}}_{2,5}(s)},$$

where $\Phi^{\mathrm{top}}_{2,5}(s) \in \mathbb{Z}[s]$ *is an explicitly determined irreducible polynomial of degree* 71 *and* $\Psi^{\mathrm{top}}_{2,5}(s)$ *is*

$$\begin{aligned}
&(38s-225)(37s-223)(35s-216)(31s-199)(31s-200)(29s-189)(29s-190)\\
&(26s-165)(25s-153)(25s-161)(25s-166)(23s-151)(23s-153)(22s-141)\\
&(22s-145)(21s-130)(20s-131)(19s-112)(19s-122)(17s-93)(17s-108)\\
&(17s-112)(17s-113)(15s-89)(14s-85)(13s-70)(13s-81)(13s-82)\\
&(13s-88)(12s-77)(11s-71)(11s-72)(10s-63)^2(9s-44)(9s-46)(9s-47)\\
&(9s-55)(9s-58)^2(9s-59)(8s-45)(8s-51)(8s-53)^2(7s-41)(7s-43)^2\\
&(7s-46)^2(6s-37)(5s-21)(5s-22)(5s-23)(5s-24)(5s-31)(5s-32)\\
&(5s-33)^2(4s-21)(4s-23)^3(4s-25)(3s-14)(3s-16)(3s-17)(3s-19)^2\\
&(3s-20)^2(2s-11)^2(2s-13)^3(s-1)(s-2)(s-3)(s-4)^2(s-6)^4 s.
\end{aligned}$$

For $d = 6$, the computation of both the $\mathfrak{p}$-adic and the topological zeta function is currently out of our reach. We record the explicit formula for the reduced zeta function, computed using Evseev's method.

Theorem 7.8 ($d = 6$, reduced; [7])**.**

$$\zeta^{\mathrm{red}}_{\mathfrak{f}_{2,6}}(t) = \frac{\Phi^{\mathrm{red}}_{2,6}(t)}{(1-t^6 1)^6(1-t^5)^6(1-t^4)^6(1-t)^3},$$

where $\Phi_{2,6}^{\mathrm{red}}(t)$ *is*

$$\begin{aligned}
&t^{72} + 3t^{71} + 36t^{70} + 145t^{69} + 669t^{68} + 2562t^{67} + 8649t^{66} + 27045t^{65} + 77670t^{64} \\
&+ 206735t^{63} + 515748t^{62} + 1211748t^{61} + 2692110t^{60} + 5682609t^{59} \\
&+ 11436687t^{58} + 22007442t^{57} + 40598238t^{56} + 71961840t^{55} + 122797673t^{54} \\
&+ 202076190t^{53} + 321171642t^{52} + 493662867t^{51} + 734688480t^{50} + 1059758436t^{49} \\
&+ 1482992565t^{48} + 2014885665t^{47} + 2659813131t^{46} + 3413604248t^{45} + 4261613451t^{44} \\
&+ 5177738109t^{43} + 6124749888t^{42} + 7056165426t^{41} + 7919643378t^{40} + 8661618634t^{39} \\
&+ 9232638888t^{38} + 9592688376t^{37} + 9715718352t^{36} + 9592688376t^{35} + 9232638888t^{34} \\
&+ 8661618634t^{33} + 7919643378t^{32} + 7056165426t^{31} + 6124749888t^{30} + 5177738109t^{29} \\
&+ 4261613451t^{28} + 3413604248t^{27} + 2659813131t^{26} + 2014885665t^{25} + 1482992565t^{24} \\
&+ 1059758436t^{23} + 734688480t^{22} + 493662867t^{21} + 321171642t^{20} + 202076190t^{19} \\
&+ 122797673t^{18} + 71961840t^{17} + 40598238t^{16} + 22007442t^{15} + 11436687t^{14} \\
&+ 5682609t^{13} + 2692110t^{12} + 1211748t^{11} + 515748t^{10} + 206735t^{9} \\
&+ 77670t^{8} + 27045t^{7} + 8649t^{6} + 2562t^{5} + 669t^{4} + 145t^{3} + 36t^{2} + 3t + 1.
\end{aligned}$$

Our computations of $\mathfrak{p}$-adic zeta functions allow us to confirm the second part of [10, Conjecture IV ($\mathfrak{P}$-adic form)] for small values of d.

Corollary 7.9. *For all* $d \in \{2,3,4\}$*, the following holds:*

$$\left.\frac{\zeta_{\mathfrak{f}_{2,d}(\mathfrak{o})}(s)}{\zeta_{\mathfrak{o}^D}(s)}\right|_{s=0} = 1,$$

where $D = d + d' = \binom{d+1}{2}$ *is the* $\mathbb{Z}$*-rank of* $\mathfrak{f}_{2,d}$.

The conjecture's first part holds for all d and all but a finite number of $q_{\mathfrak{o}}$, see Theorem 5.13.

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