Ask zeta functions of joins of graphs

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In previous work [18], we studied rational generating functions ("ask zeta functions") associated with graphs and hypergraphs. These functions encode average sizes of kernels of generic matrices with support constraints determined by the graph or hypergraph in question, with applications to the enumeration of linear orbits and conjugacy classes of unipotent groups.

In the present article, we turn to the effect of a natural graph-theoretic operation on associated ask zeta functions. Specifically, we show that two instances of rational functions, $W^-_\Gamma(X,T)$ and $W^\sharp_\Gamma(X,T)$, associated with a graph Γ are both well-behaved under taking joins of graphs. In the former case, this has applications to zeta functions enumerating conjugacy classes associated with so-called graphical groups.

Contents

1 Introduction

Ask zeta functions encode average kernel sizes within families of linear maps over finite quotients of infinite rings. Matrices of linear forms lead, via specialisation of variables,

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to associated ask zeta functions. Such zeta functions first arose by linearising the enumeration of linear orbits and conjugacy classes of groups. Their rich algebraic and combinatorial structure established them as objects of independent interest.

Combinatorial incidence structures such as graphs and hypergraphs lead to matrices of linear forms by "linearising" their adjacency and incidence matrices. In [18], we initiated the study of such (hyper)graphical ask zeta functions over compact discrete valuation rings. In particular, for each graph Γ, we showed that there exist *rational* generating functions $W_\Gamma^+(X,T)$ and $W_\Gamma^-(X,T)$ that capture average kernel sizes associated with the symmetric and antisymmetric linearised adjacency matrix of Γ, respectively.

In the present paper, we develop new tools for studying the functions $W_\Gamma^\pm$, allowing us to contribute to two contemporary research themes in the area: rigidity of zeta functions and effects of operations on zeta functions. Rigidity is an umbrella term for the phenomenon that certain algebraic or combinatorial operations on matrices of linear forms leave associated ask zeta functions unchanged. Here we show specifically that for reflexive graphs Γ, three a priori quite different ask zeta functions W_Γ^+, W_Γ^-, and $W_{\mathscr{A}\!dj(\Gamma)}$ all coincide. The effects of natural— and seemingly innocuous—operations on zeta functions associated with algebraic structures are generally mysterious and poorly understand. We show that $\Gamma \rightsquigarrow W_\Gamma^-$ is well-behaved under joins for *all* loopless graphs (answering a question from [18]) and that $\Gamma \rightsquigarrow W_\Gamma^+$ is well-behaved under joins for arbitrary *reflexive* graphs: in both cases we give concise formulae for the zeta functions of the joins in terms of the zeta functions of the graphs being joined, vastly generalising results from [18]. Our results have applications to class-counting zeta functions of so-called graphical groups.

1.1 Ask zeta functions derived from matrices of linear forms

We briefly recall selected concepts and results pertaining to ask zeta functions. For further details and references, see §2.

From matrices of linear forms to ask zeta functions. Let $\mathfrak{O}$ be a compact discrete valuation ring (DVR) with maximal ideal $\mathfrak{P}$. Examples include the ring $\mathbf{Z}_p$ of p-adic integers for a prime p and the ring $\mathbf{F}_q[\![z]\!]$ of formal power series over a finite field $\mathbf{F}_q$. Let $A = A(X_1,\dots,X_\ell) \in \mathrm{M}_{d\times e}(\mathfrak{O}[X_1,\dots,X_\ell])$ be a matrix of linear forms. The **(algebraic) ask zeta function** of A over $\mathfrak{O}$ is the formal power series $\mathsf{Z}^{\mathrm{ask}}_{A/\mathfrak{O}}(T) = \sum_{k=0}^{\infty} \alpha_k T^k \in \mathbf{Q}[\![T]\!]$ defined as follows. Given $k \geqslant 0$, by specialising variables, each $x \in (\mathfrak{O}/\mathfrak{P}^k)^\ell$ gives rise to a module homomorphism $A(x)\colon (\mathfrak{O}/\mathfrak{P}^k)^n \to (\mathfrak{O}/\mathfrak{P}^k)^m$ (acting by right multiplication on rows). The coefficient $\alpha_k \in \mathbf{Q}$ is the average size of the kernel among the maps $A(x)$ as x ranges over $(\mathfrak{O}/\mathfrak{P}^k)^\ell$. If $\mathfrak{O}$ has characteristic zero, then $\mathsf{Z}^{\mathrm{ask}}_{A/\mathfrak{O}}(T) \in \mathbf{Q}(T)$ by [13, Thm 1.4]. Ask zeta functions were introduced in [13] as linearisations of zeta functions enumerating linear orbits and conjugacy classes of suitable groups.

Global templates for ask zeta functions and arithmetic questions. Given a matrix of linear forms A over $\mathbf{Z}$, we may consider the ask zeta function $\mathsf{Z}^{\mathrm{ask}}_{A/\mathfrak{O}}(T)$ for each compact DVR $\mathfrak{O}$. It is then natural to ask how these ask zeta functions vary with $\mathfrak{O}$. Let q be the size of the residue field of $\mathfrak{O}$. Excluding finitely many exceptional residue characteristics, general machinery from p-adic integration [13, Thm 4.11–4.12] provides a "Denef formula" for $\mathsf{Z}^{\mathrm{ask}}_{A/\mathfrak{O}}(T)$ in terms of rational functions in q and T and the numbers of rational points of certain schemes over $\mathbf{Z}$ over the residue field of $\mathfrak{O}$. The presence of numbers of rational points on schemes turns out to be unavoidable in a precise sense, see [17]. In general, one can therefore expect the study of ask zeta functions $\mathsf{Z}^{\mathrm{ask}}_{A/\mathfrak{O}}(T)$ for a fixed $\mathbf{Z}$-defined A and varying $\mathfrak{O}$ to involve difficult arithmetic problems.

1.2 Matrices of linear forms from graphs and hypergraphs

Two types of matrices of linear forms are of particular interest in the present paper.

Matrices of linear forms from hypergraphs. By a **hypergraph**, we mean a triple $\mathsf{H} = (V, E, \imath)$ where V and E are finite sets of **vertices** and **hyperedges**, respectively, and $\imath \subset V \times E$ is the **incidence relation** of H. Hypergraphs are also referred to as **incidence structures** in the literature. The relation $\imath$ can be equivalently described by the **support function** $\|\cdot\| = \|\cdot\|_{\mathsf{H}} \colon E \to 2^V$ given by $\|e\| = \{v \in V : v \,\imath\, e\}$ as in [18].

Let $\mathsf{H} = (V, E, \imath)$ be a hypergraph with distinct vertices $v_1, \dots, v_n$ and distinct hyperedges $e_1, \dots, e_m$. Up to isomorphism (suitably defined, see §3.1), H is completely determined by the $n \times m$ **incidence matrix** whose (i, j)-entry is 1 if $v_i \,\imath\, e_j$ and 0 otherwise. Let $\mathsf{A}_{\mathsf{H}} = [X_{ij}]$ be the $n \times m$ matrix of linear forms over $\mathbf{Z}$ such that $X_{ij} = 0$ if and only if $v_i \not\imath\, e_j$ and such that the nonzero X_{ij} are algebraically independent over $\mathbf{Z}$. The matrix A_{H} is well-defined up to suitable equivalence, see §2. Note that $\mathsf{A}_{\mathsf{H}}(1, \dots, 1)$ is the incidence matrix of H as defined above.

Example 1.1. Consider the hypergraph H with vertices $v_1, \dots, v_4$, hyperedges $e_1, \dots, e_4$, and associated incidence matrix

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

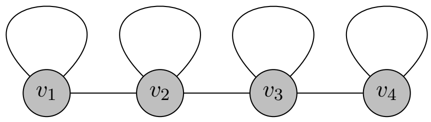
Hence, $\|e_1\| = \{v_1, v_2\}$, $\|e_2\| = \{v_1, v_2, v_3\}$, $\|e_3\| = \{v_2, v_3, v_4\}$, $\|e_4\| = \{v_3, v_4\}$, and

$$\mathsf{A}_{\mathsf{H}} = \begin{bmatrix} X_{11} & X_{12} & 0 & 0 \\ X_{21} & X_{22} & X_{23} & 0 \\ 0 & X_{32} & X_{33} & X_{34} \\ 0 & 0 & X_{43} & X_{44} \end{bmatrix}.$$

Matrices of linear forms from graphs. Let Γ be a graph with distinct vertices $v_1, \dots, v_n$ and adjacency relation $\sim$. In this article, unless otherwise indicated, graphs are allowed to contain loops (i.e. $v_i \sim v_i$ is possible) but no parallel edges. Let Γ have m edges. We define matrices of linear forms A^+_Γ and A^-_Γ over $\mathbf{Z}$ defined as follows. Let X_{ij} denote the (i, j)-entry of $\mathsf{A}^\pm_\Gamma$. We impose the following conditions:

- $X_{ij} = 0$ if and only if $v_i \not\sim v_j$,
- $X_{ji} = \pm X_{ij}$ whenever $i \neq j$, and
- The nonzero X_{ij} with $i \leqslant j$ are algebraically independent over $\mathbf{Z}$.

Example 1.2. For the graph Γ given by



we obtain

$$\mathsf{A}_\Gamma^\pm = \begin{bmatrix} X_{11} & X_{12} & 0 & 0 \\ \pm X_{12} & X_{22} & X_{23} & 0 \\ 0 & \pm X_{23} & X_{33} & X_{34} \\ 0 & 0 & \pm X_{34} & X_{44} \end{bmatrix}.$$

As in the case of hypergraphs, the matrix $\mathsf{A}_\Gamma^\pm$ does not merely depend on Γ but also on our choice of a total order on its vertices. Again, different choices yield equivalent matrices of linear forms (see §2). The matrix $\mathsf{A}_\Gamma^+(1,\dots,1)$ is the usual adjacency matrix of Γ relative to the given order of the vertices.

1.3 Background: the Uniformity Theorem

Belkale and Brosnan [1] showed that counting $\mathbf{F}_q$-rational points on the degeneracy loci of the matrices A_Γ^+ is, in a precise sense, as hard as counting $\mathbf{F}_q$-rational points on arbitrary schemes. Average sizes of kernels are expressible in terms of the numbers of matrices of given rank (see [13, §2.1]). This notwithstanding, the following result shows that at least for the matrices A_H and $\mathsf{A}_\Gamma^\pm$, such hard geometric problems average out when passing to ask zeta functions.

Theorem 1.3 (Uniformity Theorem [18, Thm A])**.**

(i) *Let* H *be a hypergraph. Then there exists* $W_\mathsf{H}(X,T) \in \mathbf{Q}(X,T)$ *such that for each compact* DVR $\mathfrak{O}$ *with residue field size* q*, we have* $\mathsf{Z}^{\mathrm{ask}}_{\mathsf{A}_\mathsf{H}/\mathfrak{O}}(T) = W_\mathsf{H}(q,T)$.

(ii) *Let* Γ *be a graph. Then there exists* $W_\Gamma^+(X,T) \in \mathbf{Q}(X,T)$ *such that for each compact* DVR $\mathfrak{O}$ *with* <u>*odd*</u> *residue field size* q*, we have* $\mathsf{Z}^{\mathrm{ask}}_{\mathsf{A}_\Gamma^+/\mathfrak{O}}(T) = W_\Gamma^+(q,T)$.

(iii) *Let* Γ *be a graph. Then there exists* $W_\Gamma^-(X,T) \in \mathbf{Q}(X,T)$ *such that for each compact* DVR $\mathfrak{O}$ *with (arbitrary) residue field size* q*, we have* $\mathsf{Z}^{\mathrm{ask}}_{\mathsf{A}_\Gamma^-/\mathfrak{O}}(T) = W_\Gamma^-(q,T)$.

Remark 1.4.

(i) In [18], Theorem 1.3(iii) was only spelled out in case Γ is loopless. However, in [18, §6], both parts (ii) and (iii) of Theorem 1.3 were proved simultaneously and, in particular, the proof via [18, Thm 6.4(iii)] also applies when Γ has a loop.

(ii) The first author's package Zeta [16] for SageMath [19] implements algorithms for computing $W^{\pm}_{\Gamma}$ and W_{H}. These algorithms are practical for small (hyper)graphs, say on at most 7 vertices.

(iii) The present article provides us with a new and self-contained proof of Theorem 1.3; see §4 and §6.5.

The present article contributes to our understanding of the operation $\Gamma \rightsquigarrow W^-_\Gamma$. In particular, we will see that for two very natural classes of graphs, namely loopless and reflexive ones, taking joins has a remarkably tame algebraic effect on W^-_Γ.

1.4 Result: joins of loopless graphs

Let Γ_1 and Γ_2 be graphs and let $\Gamma_1 \oplus \Gamma_2$ denote their disjoint union. The **join** $\Gamma_1 \vee \Gamma_2$ of Γ_1 and Γ_2 is obtained from $\Gamma_1 \oplus \Gamma_2$ by adding edges connecting each vertex of Γ_1 to each vertex of Γ_2. Note that a join of loopless graphs is again loopless.

Theorem A (Loopless joins)**.** *Let Γ_1 and Γ_2 be loopless graphs on n_1 and n_2 vertices, respectively. Write $z_i = X^{-n_i}$. Then*

$$W^-_{\Gamma_1\vee\Gamma_2}(X,T) = \frac{z_1z_2XT-1+W^-_{\Gamma_1}(X,z_2T)(1-z_2T)(1-z_2XT)+W^-_{\Gamma_2}(X,z_1T)(1-z_1T)(1-z_1XT)}{(1-T)(1-XT)}. \tag{1.1}$$

Remark 1.5 (Precursors to Theorem A)**.**

(i) In the special case that Γ_1 and Γ_2 are *cographs*, i.e. loopless graphs which do not contain a path on four vertices as induced subgraphs, Theorem A was first proved in [18, Prop. 8.4]. That proof relied on the Cograph Modelling Theorem (Theorem 1.15 below) from [18]. Theorem A thus provides a positive answer to [18, Question 10.1].

Removing the assumption that Γ_1 and Γ_2 be cographs constitutes a significant extension of the scope of this result. For instance, it is well known that asymptotically, the number of (isomorphism classes of) loopless graphs on n vertices grows like $\gamma_n := 2^{\binom{n}{2}}/n!$; see [9, §9.1]. Ravelomanana and Thimonier [11, Thm 4] showed that asymptotically, the number of (isomorphism classes of) cographs on n vertices grows like $\beta_n := C\alpha^n/n^{3/2}$, where $C = 0.206\ldots$ and $\alpha = 3.560\ldots$. In particular, β_n grows at most exponentially while γ_n grows super-exponentially as $n \to \infty$.

(ii) By [15, Prop. 8.5], equation (1.1) holds modulo T^2. Given a graph Γ, the coefficient of T of $W^-_\Gamma(X,T)$ encodes the average size of the kernel of the matrices A^-_Γ over finite fields.

Remark 1.6.

(i) The conclusion of Theorem A does not generally hold unless Γ_1 and Γ_2 are *both* loopless. Taking Γ_1 to be an isolated vertex and Γ_2 to be a loop at one vertex provides a counterexample. On the other hand, when Γ_1 and Γ_2 contain loops at all vertices, then $W^-_{\Gamma_1 \vee \Gamma_2}$ is again expressible in terms of $W^-_{\gamma_1}$ and $W^-_{\Gamma_2}$ albeit in terms of a formula other than (1.1); see Proposition C.

(ii) Taking Γ_1 to be a singleton and Γ_2 to be a path on 2 vertices (so that $\Gamma_1 \vee \Gamma_2$ is a triangle) shows that the conclusion of Theorem A does not carry over to $W^+_{\Gamma_1 \vee \Gamma_2}(X, T)$. (See [18, Table 1].)

Enter $W^\flat_\Gamma$: renormalising W^-_Γ. Theorem A may be rephrased as follows. For a graph Γ on n vertices, let $W^\flat_\Gamma = W^\flat_\Gamma(X, T) = W^-_\Gamma(X, X^n T)$. Of course, W^-_Γ and $W^\flat_\Gamma$ determine one another for given n. While the coefficients of $W^-_\Gamma(X, T)$ as a series in T encode average sizes of kernels of specialisations of $\mathsf{A}^-_\Gamma(X)$ over finite quotients of DVRs, the coefficients of $W^\flat_\Gamma(X, T)$ count pairs (v, x) with $v\mathsf{A}^-_\Gamma(x) = 0$ over such rings.

Let Γ_1 and Γ_2 be graphs on n_1 and n_2 vertices, respectively. If Γ_1 and Γ_2 are both loopless, then Theorem A takes the following form:

$$W^\flat_{\Gamma_1 \vee \Gamma_2}(X, T) = \frac{XT - 1 + W^\flat_{\Gamma_1} \cdot (1 - X^{n_1}T)(1 - X^{n_1+1}T) + W^\flat_{\Gamma_2} \cdot (1 - X^{n_2}T)(1 - X^{n_2+1}T)}{(1 - X^{n_1+n_2}T)(1 - X^{n_1+n_2+1}T)}. \tag{1.2}$$

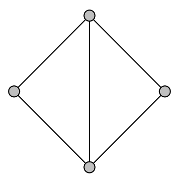
Join powers. Noting that taking joins of graphs is an associative operation (up to isomorphism), let $\Gamma^{\vee k} = \Gamma \vee \cdots \vee \Gamma$, with k copies of Γ. We let $\Gamma^{\vee 0}$ denote the graph without vertices, the identity element with respect to $\vee$. Theorem A provides us with infinitely many *explicit* formulae for the functions $W^\flat_\Gamma$ (hence also W^-_Γ):

Corollary 1.7. *Let Γ be a loopless graph on n vertices. Let $k \geqslant 0$. Then*

$$W^\flat_{\Gamma^{\vee k}}(X, T) = \frac{(k-1)(XT - 1) + kW^\flat_\Gamma \cdot (1 - X^n T)(1 - X^{n+1}T)}{(1 - X^{kn}T)(1 - X^{kn+1}T)}.$$

Proof. Induction on k using $\Gamma^{\vee(k+1)} \approx \Gamma \vee \Gamma^{\vee k}$ and (1.2). ♦

Example 1.8. Let Γ be the following graph:



By [18, Thm 8.18] or [18, Table 1], we have $W^-_\Gamma = \frac{(1-X^{-1}T)(1-X^{-2}T)}{(1-T)^2(1-XT)}$ and thus $W^\flat_\Gamma = W^-_\Gamma(X, X^4T) = \frac{(1-X^2T)(1-X^3T)}{(1-X^4T)^2(1-X^5T)}$. Using Corollary 1.7, we find that

$$W^\flat_{\Gamma^{\vee k}}(X, T) = \frac{X^5T^2 + (k-1)(X^4T + XT) - k(X^3T + X^2T) + 1}{(1 - X^4T)(1 - X^{4k}T)(1 - X^{4k+1}T)}$$

and thus

$$\begin{aligned}W^-_{\Gamma^{\vee k}}(X,T) &= W^\flat_\Gamma(X, X^{-4k}T)\\ &= \frac{X^{5-8k}T^2 + (k-1)(X^{4-4k}T + X^{1-4k}T) - k(X^{3-4k}T + X^{2-4k}T) + 1}{(1-X^{4-4k}T)(1-T)(1-XT)}.\end{aligned}$$

1.5 Related work: disjoint unions and Hadamard products

Theorem A shows that one natural graph-theoretic operation, taking joins, has a transparent effect on the rational functions W^-_Γ attached to loopless graphs. There are of course many other natural binary operations on (loopless) graphs that one may consider. To our knowledge, only one of these has been investigated so far: disjoint unions.

Recall that given formal power series $F(T) = \sum_{k=0}^\infty a_k T^k$ and $G(T) = \sum_{k=0}^\infty b_k T^k$ with coefficients in a field k, their **Hadamard product** is $F(T) *_T G(T) = \sum_{k=0}^\infty a_k b_k T^k$. It is well known that if $F(T), G(T) \in k(T)$, then $F(T) *_T G(T) \in k(T)$; see e.g. [2, Ch. 1]. It is easy to see that for each graph Γ, we have $W^\pm_{\Gamma_1 \oplus \Gamma_2}(X,T) = W^\pm_{\Gamma_1}(X,T) *_T W^\pm_{\Gamma_2}(X,T)$; cf. [18, §8.2]. Here, the Hadamard product is taken over $k = \mathbf{Q}(X)$. We note that $W^\flat_{\Gamma_1\oplus\Gamma_2} = W^\flat_{\Gamma_1} *_T W^\flat_{\Gamma_2}$ follows easily; cf. [5, Lem. 5.9].

In general, explicitly expressing Hadamard products of rational generating functions as (explicit sums of) rational functions appears to be difficult. A growing body of research has provided algebro-combinatorial tools for studying Hadamard products of ask zeta functions (in particular those associated with graphs), orbit-counting, and class-counting zeta functions in fortunate cases, see [13, §2.3], [18, §5.2], and [6] (see also [5]). We record some consequences of these results in §12.

1.6 Result: the Reflexive Graph Modelling Theorem

Let Γ be a graph with vertex set V and adjacency relation $\sim$ on V. Anticipating a definition from §3.3, we write $\mathscr{A}\!dj(\Gamma)$ for the **adjacency hypergraph** $(V, V, \sim)$ of Γ. Hence, $\mathscr{A}\!dj(\Gamma)$ is the hypergraph obtained by viewing an adjacency matrix of Γ as incidence matrix of a hypergraph. Recall that Γ is **reflexive** if $v \sim v$ for each $v \in V$. The following is our second main result.

Theorem B (Reflexive Graph Modelling Theorem)**.** *Let Γ be a reflexive graph. Then*

$$W^+_\Gamma = W^-_\Gamma = W_{\mathscr{A}\!dj(\Gamma)}.$$

Example 1.9. Let H be as in Example 1.1 and Γ be as in Example 1.2. Then Theorem B shows that $W^\pm_\Gamma = W_{\mathsf{H}}$. Using Zeta [16], we find this common rational function to be

$$\frac{1 + 2X^{-1}T - X^{-2}T^2 - 6X^{-2}T + 6X^{-4}T^2 + X^{-4}T - 2X^{-5}T^2 - X^{-6}T^3}{(1-X^{-1}T)^2(1-T)^2}.$$

We offer three perspectives on Theorem B. The first portrays it as a new "modelling theorem", the second as a new rigidity result, and the third—developed in Section 1.7—views it in the context of reflexive joins.

A new modelling theorem. First, Theorem B is a new "modelling theorem" that belongs to the same genre as the Cograph Modelling Theorem [18, Thm D] (Theorem 1.15 below). Indeed, it asserts that for each graph Γ of a certain type (namely, each reflexive graph), there exists a modelling hypergraph $\mathsf{H} = \mathscr{Adj}(\Gamma)$ such that $W_\Gamma^\pm = W_\mathsf{H}$. Such a result allows us to leverage what is known about the rational functions W_H; see §1.9 below.

Rigidity of ask zeta functions. Second, we may view Theorem B as a new instance of rigidity phenomena that have been previously explored in the study of ask zeta functions. The first such result is [13, Cor. 5.10]. It asserts that for any $d > 1$ and each compact DVR $\mathfrak{O}$ with residue field size q, the ask zeta functions associated with the generic $d \times d$ matrix A_d and the generic traceless $d \times d$ matrix T_d over $\mathfrak{O}$ coincide; this common ask zeta function is $(1 - q^{-d}T)/(1 - T)^2$. That is, imposing the linear relation $\mathrm{trace}(A_d) = 0$ on the entries of A_d has no effect on associated ask zeta functions. This result was significantly extended in [7, Thm A], which showed that imposing suitably *admissible* systems of linear equations involving the entries of generic rectangular, symmetric, or antisymmetric matrices has no effects on ask zeta functions.

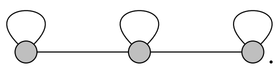
Theorem B is a new result of this form. Indeed, given a reflexive graph Γ, the matrix $\mathsf{A}_\Gamma^\pm = [a_{ij}]$ is obtained from $\mathsf{A}_{\mathscr{Adj}(\Gamma)}$ by imposing the linear relations $a_{ij} = \pm a_{ji}$ for $i \neq j$. Crucially, these relations are *not* among those covered by [7, Thm A].

Viewing $\mathsf{A}_\Gamma^\pm$ as being obtained from $\mathsf{A}_{\mathscr{Adj}(\Gamma)}$ by imposing off-diagonal (anti)symmetry relations, it is natural ask whether the same conclusion holds for more general classes of matrices of linear forms. Indeed, even slight generalisations of the matrices $\mathsf{A}_{\mathscr{Adj}(\Gamma)}$ yield examples for which natural analogues of the conclusions of Theorem B no longer hold.

Example 1.10. Consider the following matrices of linear forms:

$$A = \begin{bmatrix} U_1 & X & 0 \\ X & U_2 & X \\ 0 & X & U_3 \end{bmatrix}, \quad B = \begin{bmatrix} U_1 & X & 0 \\ Y & U_2 & X \\ 0 & Y & U_3 \end{bmatrix}, \quad C = \begin{bmatrix} U_1 & X & 0 \\ Y & U_2 & X \\ 0 & Z & U_3 \end{bmatrix}.$$

Using Zeta [16], we find that for almost all residue characteristics of $\mathfrak{O}$, the zeta functions $\mathsf{Z}^{\mathrm{ask}}_{A/\mathfrak{O}}$, $\mathsf{Z}^{\mathrm{ask}}_{B/\mathfrak{O}}$, and $\mathsf{Z}^{\mathrm{ask}}_{C/\mathfrak{O}}$ are all distinct. They also all differ from $\mathsf{Z}^{\mathrm{ask}}_{\mathsf{A}_\Gamma^\pm/\mathfrak{O}}$, where Γ is the "looped path"

.

Using the notation from the next subsection, a formula for the latter zeta function is recorded in Table 2 in the row corresponding to the (simple) path on three vertices.

1.7 A new graph invariant ($W_\Gamma^\sharp$) and reflexive joins

Given a graph Γ, let $\hat{\Gamma}$ denote its reflexive closure, obtained by adding all missing loops. For our third perspective on Theorem B, observe that $\Gamma \mapsto \hat{\Gamma}$ yields a bijection between loopless and reflexive graphs. Theorem B therefore suggests the study of the following rational functions attached to loopless graphs.

Definition 1.11. For a graph Γ, let $W^\sharp_\Gamma(X,T)$ denote the common value of

$$W^+_{\hat{\Gamma}}(X,T) = W^-_{\hat{\Gamma}}(X,T) = W_{\mathcal{A}dj(\hat{\Gamma})}(X,T).$$

For a list of all $W^\sharp_\Gamma$ as Γ ranges over graphs on at most four vertices, see Table 2. The $W^\sharp_\Gamma$ appear to be remarkably well-behaved. In particular, using Theorem B and results from [18], we obtain the following reflexive counterpart of Theorem A.

Proposition C (Reflexive joins and disjoint unions)**.** *Let Γ_1 and Γ_2 be (loopless) graphs on n_1 and n_2 vertices, respectively. Write $z_i = X^{-n_i}$. Then $W^\sharp_{\Gamma_1\oplus\Gamma_2} = W^\sharp_{\Gamma_1} *_T W^\sharp_{\Gamma_2}$ and*

$$W^\sharp_{\Gamma_1\vee\Gamma_2}(X,T) = \frac{z_1z_2T - 1 + W^\sharp_{\Gamma_1}(X,z_2T)(1-z_2T)^2 + W^\sharp_{\Gamma_2}(X,z_1T)(1-z_1T)^2}{(1-T)^2}.$$

1.8 Group-theoretic context: graphical groups and their conjugacy classes

As shown in [13, 14], ask zeta functions arise naturally in the enumeration of linear orbits and conjugacy classes of unipotent groups. We recall the connection between ask and class-counting zeta functions in the special case of Baer group schemes.

Class-counting zeta functions of Baer group schemes. Let $\mathrm{k}(H)$ denote the number of conjugacy classes of a group H. Let $\mathbf{G}$ be a group scheme of finite type over $\mathfrak{O}$. Inspired by [8], the **class-counting zeta function** of $\mathbf{G}$ is $\mathsf{Z}^{\mathrm{cc}}_{\mathbf{G}}(T) = \sum_{k=0}^{\infty} \mathrm{k}(\mathbf{G}(\mathfrak{O}/\mathfrak{P}^k))T^k$.

Let $A = A(X_1,\dots,X_\ell) \in \mathrm{M}_d(\mathbf{Z}[X_1,\dots,X_\ell])$ be an antisymmetric matrix of linear forms. We identify linear forms in $\mathbf{Z}[X_1,\dots,X_\ell]$ and elements of $\mathbf{Z}^\ell$ with X_i corresponding to the ith standard basis vector of $\mathbf{Z}^\ell$. With this identification, A is equivalently described by the alternating bilinear product $\diamond\colon \mathbf{Z}^d \times \mathbf{Z}^d \to \mathbf{Z}^\ell$ given by $x \diamond y = xAy^\top$ $(x, y \in \mathbf{Z}^d)$. Using a geometric variant from [21, §2.4] of the classical Baer correspondence, the **Baer group scheme** $\mathbf{G}_\diamond$ attached to $\diamond$, and hence to A, was defined in [18, §2.4]. The following was proved in [18] in the special case that the map $x \mapsto A(x)$ on $\mathbf{Z}^\ell$ is injective—the same arguments apply without this assumption.

Proposition 1.12 (Cf. [18, Prop. 1.1])**.** *Let $A \in \mathrm{M}_d(\mathbf{Z}[X_1,\dots,X_\ell])$ be an antisymmetric matrix of linear forms. Let $\diamond\colon \mathbf{Z}^d \times \mathbf{Z}^d \to \mathbf{Z}^\ell$ be the alternating bilinear product attached to A as above. Let $\mathfrak{O}$ be a compact* DVR *of arbitrary characteristic. Let q be the size of the residue field of $\mathfrak{O}$. Then $\mathsf{Z}^{\mathrm{cc}}_{\mathbf{G}_\diamond\otimes\mathfrak{O}}(T) = \mathsf{Z}^{\mathrm{ask}}_{A/\mathfrak{O}}(q^\ell T)$.*

Graphical groups and their class-counting zeta functions. Let Γ be a loopless graph. The **graphical group scheme** $\mathbf{G}_\Gamma$ (over $\mathbf{Z}$) associated with Γ was defined in [18, §3.4]. For a group-theoretic description, see [15, §1.1]. Using our notation from §§1.1–1.2, $\mathbf{G}_\Gamma$ is the Baer group scheme associated with the alternating bilinear product attached to the antisymmetric matrix of linear forms A^-_Γ. (We require Γ to be loopless for A^-_Γ to be antisymmetric.) The group $\mathbf{G}_\Gamma(\mathbf{Z})$ is the maximal nilpotent quotient of class at most two of the right-angled Artin group associated with the (loopless) complement of Γ. For each odd prime p, we have $\mathbf{G}_\Gamma(\mathbf{F}_p) \approx \mathbf{G}_\Gamma(\mathbf{Z})/\mathbf{G}_\Gamma(\mathbf{Z})^p$. The following consequence of Proposition 1.12 provides a group-theoretic motivation for studying the functions W^-_Γ.

Corollary 1.13 (Cf. [18, Prop. 3.9])**.** *Let* Γ *be a loopless graph with* m *edges. Let* $\mathfrak{O}$ *be a compact* DVR *with residue field of size* q*. Then* $\mathsf{Z}^{\mathrm{cc}}_{\mathbf{G}_\Gamma\otimes\mathfrak{O}}(T) = W^-_\Gamma(q, q^m T)$*.* ♦

In this way, our results from §1.4 have immediate group-theoretic consequences in that they provide us with formulae for class-counting zeta functions of graphical groups.

1.9 Related work: from hypergraphs to cographs and back again

Theorem 1.3(ii)–(iii) notwithstanding, it appears to be very difficult to produce explicit examples of the rational functions $W^\pm_\Gamma(X,T)$ for interesting families of graphs. On the other hand, we do have a very precise formula for $W_{\mathsf{H}}(X,T)$. As in [18], for a finite set V, let $\widehat{\mathrm{WO}}(V)$ denote the poset of (possibly empty) flags of (possibly empty) subsets of V.

Theorem 1.14 (Cf. [18, Thm C])**.** *Let* $\mathsf{H} = (V, E, \imath)$ *be a hypergraph. For* $U \subset V$*, define* $\check{U} = \{e \in E : \exists u \in U, u \,\imath\, e\}$*. Then*

$$W_{\mathsf{H}}(X,T) = \sum_{y\in\widehat{\mathrm{WO}}(V)} (1-X^{-1})^{|\mathrm{sup}(y)|} \prod_{U\in y} \frac{X^{|U|-|\check{U}|}T}{1-X^{|U|-|\check{U}|}T}. \tag{1.3}$$

Proof. For $I \subset V$, define $\mu_I = \#\{e \in E : I = \{v \in V : v \,\imath\, e\}\}$. By [18, Thm C],

$$W_{\mathsf{H}}(X,T) = \sum_{y\in\widehat{\mathrm{WO}}(V)} (1-X^{-1})^{|\mathrm{sup}(y)|} \prod_{U\in y} \frac{X^{|U|-\sum_{I\cap U\neq\emptyset}\mu_I}T}{1-X^{|U|-\sum_{I\cap U\neq\emptyset}\mu_I}T}.$$

The claim follows since for each $U \subset V$, we have $\sum\limits_{I\cap U\neq\emptyset} \mu_I = |\check{U}|$. ♦

The number of summands in (1.3) grows super-exponentially with $|V|$; see [18, (1.4)]. While Theorem 1.14 is therefore of limited use when it comes to explicitly computing $W_{\mathsf{H}}(X,T)$, it turns out to be a powerful theoretical tool; see [18, Thms E–F]. The following result from [18] constitutes a bridge between incidence structures in hypergraphs and adjacency structures in graphs.

Theorem 1.15 (Cograph Modelling Theorem [18, Thm D])**.** *Let* Γ *be a cograph. Then there exists an explicit hypergraph* H *such that* $W^-_\Gamma(X,T) = W_{\mathsf{H}}(X,T)$*.*

Hence, if Γ is a cograph, then Theorem 1.14 and its many consequences in [18] apply to $W^-_\Gamma(X,T)$. Following [18], we refer to H as in Theorem 1.15 as a **model** or a **modelling hypergraph** of Γ. Theorem B shows that if Γ is a reflexive graph, then $W^-_\Gamma(X,T) = W_{\mathsf{H}}(X,T)$ for $\mathsf{H} = \mathscr{A}\!dj(\Gamma)$. In contrast, the explicit modelling hypergraph in the proof of Theorem 1.15 in [18] is constructed recursively in terms of decompositions of a cograph Γ into joins and disjoint unions of subgraphs. We note that by combining Theorem A and results from [18, §5], we obtain a new simple new proof of Theorem 1.15 in §11.6.

1.10 Methodology

All main results in the present paper rely on a new proof of Theorem 1.3.

The Uniformity Theorem: behind the scenes. Given a matrix of linear forms A over a compact DVR $\mathfrak{O}$ with maximal ideal $\mathfrak{P}$, [13, §4] provides formulae for $\mathsf{Z}^{\mathrm{ask}}_{A/\mathfrak{O}}(T)$ in terms of $\mathfrak{P}$-adic integrals. Using ideas from [22], these integrals are expressible in terms of the ideals of minors of A itself or of one of its *Knuth duals* in the sense of [14]. In the cases of the matrices A_{H} (resp. $\mathsf{A}_{\Gamma}^{\pm}$) from §1.2, we record descriptions of such $\mathfrak{P}$-adic integrals representing the associated ask zeta functions in Proposition 3.2 (resp. Proposition 3.4) below. For the integral notation that we use, see (3.1)–(3.2). In these integrals, $\mathfrak{I}_i\mathsf{H}$ (resp. $\mathfrak{I}_i\Gamma^{\pm}$) denotes the ideal generated by the $i \times i$ minors of a certain matrix of linear forms C_{H} (resp. $\mathsf{C}_{\Gamma}^{\pm}$). In the language of [14], C_{H} (resp. $\mathsf{C}_{\Gamma}^{\pm}$) is the $\circ$-dual of A_{H} (resp. $\mathsf{A}_{\Gamma}^{\pm}$).

The proof of Theorem 1.3 (parts (ii)–(iii), in particular) in [18] was based on an analysis of integrals as in (3.1)–(3.2) using toric geometry and an elaborate recursion. Working directly with ideals of minors of matrices of linear forms can quickly become daunting and the recursive approach from [18] allowed us to completely avoid investigating any minors. In essence, for a given graph Γ, the proof of Theorem 1.3(ii)–(iii) expressed each integral (3.2) as an unspecified finite sum of *monomial* $\mathfrak{P}$-adic integrals. For each of the latter integrals, conclusions analogous to Theorem 1.3 are well known to hold; see Proposition 2.3 below. The proof of Theorem 1.15 relied on very similar core ingredients.

Key new tool: an explicit combinatorial parameterisation of minors The technical innovation of the present article is an explicit combinatorial parameterisation of the nonzero minors in the integrals (3.1)–(3.2). Our parameterisation will be obtained in Proposition 4.1 for hypergraphs and in Theorem 6.1 for graphs, the latter case being much more involved. Our parameterisation shows that (assuming invertibility of 2 in the study of A_{Γ}^{+}) each of the ideals of minors in (3.1)–(3.2) is generated by monomials. As a first application of our parameterisation, we obtain a new proof of Theorem 1.3 using the well-known uniform rationality of monomial $\mathfrak{P}$-adic integrals (Proposition 2.3).

Crucially, we prove significantly more than monomiality of the aforementioned ideals of minors. Namely, we show that up to signs (and multiplication by powers of 2 in the case of A_{Γ}^{+}), the nonzero minors in question are explicit monomials derived from combinatorial gadgets attached to hypergraphs and graphs that we call *selectors* and *animations*, respectively. Here, a **selector** of a hypergraph $\mathsf{H} = (V, E, \imath)$ is a partial function ϕ defined on some subset of E such that, whenever it is defined, ϕ sends a hyperedge e to one of its incident vertices. Similarly, an **animation** of a graph is a partial function on the vertex set which, whenever defined, sends a vertex to one of its neighbours. For formal definitions, see §4 and §6.2.

By studying algebraic and combinatorial features of animations, we obtain our two main results, Theorem A and Theorem B. When studying the rational functions $W_{\Gamma}^{-}(X,T)$ attached to loopless graph, our parameterisation of minors leads us to consider *nilpotent* animations, within an ambient monoid of partial functions. These have a rich algebraic and combinatorial structure which forms the basis of our proof of Theorem A.

1.11 Overview

In §2, we collect basic material on ask zeta functions. In §3, we review facts on ask zeta functions attached to graphs and hypergraphs. Ask zeta functions associated with hypergraphs are the subject of §4, culminating in a new proof of Theorem 1.3(i) by means of selectors. As indicated above, the case of graphs is considerably more complicated. In §5, we lay the foundation for our analysis of the rational functions $W_\Gamma^\pm$ by means of animations in §6. As a by-product, we obtain a new proof of Theorem 1.3(ii)–(iii). Drawing upon the machinery that we developed, the Reflexive Graph Modelling Theorem (Theorem B) follows quite easily in §7. The next sections lay the groundwork for the proof of Theorem A in §9. Section 8 is devoted entirely to nilpotent animations. These play a crucial role in our study of W_Γ^- for loopless Γ. In §9, we relate the nilpotent animations of a join $\Gamma_1 \vee \Gamma_2$ of two loopless graphs to those of the Γ_i. At first glance, §10 might be mistaken for a non sequitur: in it, we investigate the effect of adding generic rows to matrices of linear forms on associated ask zeta functions. This investigation will play a small but pivotal role in our proof of Theorem A in §11. Finally, in §12, we have a closer look at the rational functions $W_\Gamma^\sharp$ from Definition 1.11. In particular, we prove Proposition C, we derive an explicit formula in the spirit of Theorem 1.14 for $W_\Gamma^\sharp$ (Proposition 12.1), we deduce key analytic properties (Proposition 12.3, and we collect several examples of these rational functions.

1.12 Notation

Sets, functions, and logic. Maps usually act on the right. We write $A \sqcup B$ for the disjoint union of the sets A and B. We write $A \subset B$ to indicate that A is a not necessarily proper subset of B. For a property P, we write $[P]$ for the **Iverson bracket**

$$[P] = \begin{cases} 1, & \text{if } P \text{ is true,} \\ 0, & \text{otherwise.} \end{cases}$$

Rings and modules. All rings are assumed to be associative, commutative, and unital. Let R be a ring. By an R**-algebra**, we mean a ring S endowed with a ring map $R \to S$. Let U be a set. Let RU denote the free R-module on U with basis $(\mathsf{b}_u)_{u\in U}$. For $x \in RU$, we define $x_u \in R$ $(u \in U)$ via $x = \sum_{u\in U} x_u \mathsf{b}_u$. For a subset $A \subset R$ (not necessarily a subring or an ideal), we occasionally write $AR = \{x \in RU : x_u \in A \text{ for all } u \in U\}$.

We write $X_U = (X_u)_{u\in U}$ for a chosen set of algebraically independent variables over R. Each $a \in \mathbf{Z}U$ gives rise to the Laurent monomial $X_U^a = \prod_{u\in U} X_u^{a_u}$. By a **monomial ideal** I of $R[X_U]$, we mean an ideal of the form $I = \langle X_U^a : a \in A\rangle$ for some $A \subseteq \mathbf{N}_0U$. It is well known that if I is a monomial ideal as before, then the set A can be chosen to be finite. By a **linear form** in $R[X_U]$, we mean a polynomial of the form $\sum_{u\in U} c_u X_u$ $(c_u \in U)$. If S is an R-algebra, then by sending $x \in SU$ to the map which evaluates polynomials in X_U at x, we obtain a canonical bijection between SU and the set of R-algebra homomorphisms $R[X_U] \to S$.

Discrete valuation rings. Throughout, $\mathfrak{O}$ denotes a compact DVR with maximal ideal $\mathfrak{P}$ and residue field $\mathfrak{O}/\mathfrak{P}$ of size q and characteristic p. For a nonzero finitely generated $\mathfrak{O}$-module M, we write $M^\times = M \setminus \mathfrak{P}M$; we also set $\{0\}^\times = \{0\}$. Let $\pi \in \mathfrak{P} \setminus \mathfrak{P}^2$ denote a fixed uniformiser. Let K be the field of fractions of $\mathfrak{O}$. Let $|\cdot|$ be the absolute value on K with $|\pi| = q^{-1}$ and let $\|\cdot\|$ denote the associated maximum norm. We write μ for the Haar measure on $\mathfrak{O}$ with total volume 1. We use the same symbol for the product measure on a free $\mathfrak{O}$-module of finite rank. We let $\nu = \nu_K$ denote the normalised (additive) valuation on K with $\nu(\pi) = 1$. For a finite set U and $x \in KU$ with $\prod_{u\in U} x_u \neq 0$, we write $\nu(x) = \sum_{u\in U} \nu(x_u)\mathsf{b}_u \in \mathbf{Z}U$.

Further notation.

Notation	comment	reference
$\mathfrak{I}_k(M)$	ideal generated by $k \times k$ minors of M	§2
$\mathsf{Z}^{\mathrm{ask}}_{A/\mathfrak{O}}$, $\zeta^{\mathrm{ask}}_{A/\mathfrak{O}}$	ask zeta functions	§2
$\mathsf{H} = (V, E, \imath)$	hypergraph	§3.1
$\|e\|, \|e\|_{\mathsf{H}}$	support of the hyperedge e	§1.2
$\mathsf{H}[V' \mid E']$	subhypergraph	§3.1
$\oplus$, $\vee$	disjoint union, join / complete union	§3.1, §3.3, §11.6
$\mathsf{A}_{\mathsf{H}}, \mathsf{C}_{\mathsf{H}}$	linearised incidence matrix, its $\circ$-dual	§3.2
$\mathfrak{I}_k\mathsf{H}$	$\mathfrak{I}_k(\mathsf{C}_{\mathsf{H}})$	§3.2
$\int_W \mathsf{H}(s)$	integral expression for $\zeta^{\mathrm{ask}}_{\mathsf{A}_{\mathsf{H}}/\mathfrak{O}}(s)$	(3.1), Proposition 3.2
$\mathsf{A}^{\pm}_{\Gamma}, \mathsf{C}^{\pm}_{\Gamma}$	linearised adjacency matrix, its $\circ$-dual	§3.4
$\mathfrak{I}^{\pm}_k\Gamma$, $\frac{1}{2}\mathfrak{I}^{+}_{\Gamma}$	$\mathfrak{I}_k(\mathsf{C}^{\pm}_{\Gamma})$ over $\mathbf{Z}$ or $\mathbf{Z}[1/2]$	§3.4
$\int_W \Gamma^{\pm}(s)$	integral expression for $\zeta^{\mathrm{ask}}_{\mathsf{A}^{\pm}_{\Gamma}/\mathfrak{O}}(s)$	(3.2), Proposition 3.4
$U_\perp$	$U \sqcup \{\perp\}$	§4
$\mathcal{D}(\phi)$	domain of definition of ϕ	§4
Y^{ϕ^*}	preimage of Y under ϕ	§4
$\deg(\phi)$	$\lvert\mathcal{D}(\phi)\rvert$	§4
$\mathrm{mon}(\phi)$	monomial associated with ϕ	§4
$\phi \upharpoonright U'$	restriction of ϕ to U'	§4
$\phi[x \leftarrow y]$	redefining ϕ	§4
$\sim$	adjacency in a graph	§3.3
$\hat{\Gamma}$	reflexive closure	§1.7
$W^{\pm}_{\Gamma}$, W_{H}	ask zeta function associated with (hyper)graph	Theorem 1.3
$W^{\sharp}_{\Gamma}$	common value of $W^{+}_{\hat{\Gamma}} = W^{-}_{\hat{\Gamma}} = W_{\mathcal{Adj}(\hat{\Gamma})}$	Definition 1.11
$\mathrm{Sel}(\mathsf{H})$	selectors	§4
$\mathcal{Adj}(\Gamma)$	adjacency hypergraph	§1.6, §3.3
$\mathcal{Inc}(\Gamma)$	incidence hypergraph	§3.3
$m^{\pm}_{\Gamma}[V' \mid E']$	minor of $\mathsf{C}^{\pm}_{\Gamma}$	§5.2, §6.3
$\mathcal{ol}(\phi)$	number of ϕ-orbits of odd length > 1	§6.1
$\mathrm{Ani}(\Gamma)$	animations	§6.2
$\mathrm{Nil}(\Gamma)$, $\mathrm{Fix}(\Gamma)$, $\mathrm{Odd}(\Gamma)$	special sets of animations	§6.2
$\preccurlyeq_\alpha$	preorder derived from a partial function	§8.1
$\mathsf{last}_\alpha(v)$	unique $\preccurlyeq_\alpha$-maximal element above v	§8.1
$\preccurlyeq_u$	partial order relative to distinguished vertex	§8.3
$\simeq$	equivalence of matrices	§10.1
$\lambda(A)$, $\lambda_i(A)$	elementary divisors	§10.2

2 Background on ask zeta functions

The following is a brief introduction to ask zeta function attached to matrices of linear forms. In terms of generality, this perspective lies between [13], which considers modules of matrices, and [14], which considers so-called module representations. Let R be a ring. Let U be a finite set. Recall that RU denotes the free R-module with basis $(\mathsf{b}_u)_{u\in U}$.

Equivalence. There is a natural action of $\mathrm{GL}(RU) \times \mathrm{GL}_n(R) \times \mathrm{GL}_m(R)$ on the R-module of linear forms within $\mathrm{M}_{n\times m}(R[X_U])$: the second and third factor act by matrix multiplication on the left and right, respectively, and $\mathrm{GL}(RU)$ acts by changing coordinates of linear forms. Two matrices of linear forms $A(X_U)$ and $B(X_U)$ in $\mathrm{M}_{n\times m}(R[X_U])$ are **equivalent** if they lie in the same orbit under this action. Equivalence in this sense corresponds to *isotopy* of module representations in [14].

Ask zeta functions. Let $A(X_U) \in \mathrm{M}_{n\times m}(R[X_U])$ be a matrix of linear forms. Let S be an R-algebra. Given $x \in SU$, we view the specialisation $A(x) \in \mathrm{M}_{n\times m}(S)$ as a linear map $S^n \to S^m$ acting by right multiplication on rows. If S is finite as a set, we write

$$\mathrm{ask}_S(A(X_U)) = \frac{1}{|SU|} \sum_{x\in SU} |\mathrm{Ker}(A(x))|$$

for the average size of the kernel of these maps.

Let $\mathfrak{O}$ be a compact DVR endowed with an R-algebra structure. Recall that $\mathfrak{P}$ denotes the maximal ideal of $\mathfrak{O}$. The **(algebraic) ask zeta function** of $A(X_U)$ over $\mathfrak{O}$ is the formal power series

$$\mathsf{Z}^{\mathrm{ask}}_{A/\mathfrak{O}}(T) = \mathsf{Z}^{\mathrm{ask}}_{A(X_U)/\mathfrak{O}}(T) = \sum_{k=0}^{\infty} \mathrm{ask}_{\mathfrak{O}/\mathfrak{P}^k}(A(X_U))T^k \in \mathbf{Q}[\![T]\!].$$

If $A(X_U)$ and $B(X_U)$ are equivalent matrices of linear forms over R, then $\mathsf{Z}_{A(X_U)/\mathfrak{O}}(T) = \mathsf{Z}_{B(X_U)/\mathfrak{O}}(T)$ for each $\mathfrak{O}$ as above. We note that if $\mathfrak{O}$ has characteristic zero, then $\mathsf{Z}_{A(X_U)/\mathfrak{O}}(T) \in \mathbf{Q}(T)$; see [13, Thm 4.10]. As explained in [13], ask zeta functions arise in the enumeration of linear orbits and conjugacy classes of unipotent groups.

Writing $q = |\mathfrak{O}/\mathfrak{P}|$ for the residue field size of $\mathfrak{O}$, we write $\zeta^{\mathrm{ask}}_{A(X_U)/\mathfrak{O}}(s) = \mathsf{Z}^{\mathrm{ask}}_{A(X_U)/\mathfrak{O}}(q^{-s})$ for the **(analytic) ask zeta function** of $A(X_U)$ over $\mathfrak{O}$. The series $\zeta^{\mathrm{ask}}_{A(X_U)/\mathfrak{O}}(s)$ converges for $\mathrm{Re}(s) > n$. Moreover, $\zeta^{\mathrm{ask}}_{A(X_U)/\mathfrak{O}}(s)$ and $\mathsf{Z}^{\mathrm{ask}}_{A(X_U)/\mathfrak{O}}(T)$ determine one another so referring to both as "the" ask zeta function of $A(X_U)$ constitutes only a minor abuse of terminology.

Duals, minors, and integrals. We will study the zeta functions $\zeta^{\mathrm{ask}}_{A(X_U)/\mathfrak{O}}(s)$ of interest to us by means of suitable $\mathfrak{P}$-adic integrals. Ignoring the harmless effect of replacing $A(X_U)$ by its transpose (see [13, Lem. 2.4]), using the "Knuth duality" operations from [14] and the machinery from [13], given $A(X_U)$, we obtain three (in general very different) formulae for $\zeta^{\mathrm{ask}}_{A(X_U)/\mathfrak{O}}(s)$ by means of $\mathfrak{P}$-adic integrals. In addition, each of these formulae admits an affine and a projective version. Our choice here is the projective form of the integral attached to the $\circ$-dual of $A(X_U)$. To describe this explicitly, let us first order the elements of U and write $U = \{u_1, \ldots, u_\ell\}$ where $\ell = |U|$. Let $V = \{v_1, \ldots, v_n\}$ be a set of cardinality n. We may characterise $A(X_U)$ and a matrix $C(X_V) \in \mathrm{M}_{\ell\times m}(R[X_V])$ via

$$A(X_U)_{ij} = \sum_{k=1}^{\ell} \alpha_{ijk} X_{u_k}, \qquad C(X_V)_{kj} = \sum_{i=1}^{n} \alpha_{ijk} X_{v_i}, \tag{2.1}$$

where $\alpha_{ijk} \in R$. We refer to $C(X_V)$ as a **$\circ$-dual** of $A(X_U)$. Our use of the indefinite article reflects the fact that $C(X_V)$ depends on the chosen total orders. Note that by construction, $A(X_U)$ is a $\circ$-dual of $C(X_V)$. (The abstract version of the $\circ$-operation in [14] is genuinely idempotent.)

Given a matrix M over a ring S and $k \geqslant 0$, we write $\mathfrak{I}_k(M)$ or $\mathfrak{I}_k(M; S)$ for the ideal of S generated by the $k \times k$ minors of M. We record the following observation.

Lemma 2.1. *Let $A(X_U) \in \mathrm{M}_{n\times m}(R[X_U])$. Let S be an R-algebra and $x \in SU$. Let $i \geqslant 0$. Then $\mathfrak{I}_i(A(x); S)$ is the ideal of S generated by the image of $\mathfrak{I}_i(A(X_U); R[X_U])$ under the specialisation map $R[X_U] \to S$ determined by x.* ♦

Let us return to our $\circ$-dual $C(X_V)$ of $A(X_U)$. Let $I_k = \mathfrak{I}_k(C(X_V); R[X_V])$. Note that I_k is generated by homogeneous elements of degree k and that $I_0 = \langle 1 \rangle = R[X_V]$.

Proposition 2.2 (Cf. [13, §§4.3–4.4])**.** *Let $\mathfrak{O}$ be a compact* DVR *endowed with an R-algebra structure. Let r be the rank of the image of $C(X_V)$ in $\mathrm{M}_{\ell\times m}(K[X_V])$ over $K(X_V)$. Then for all $s \in \mathbf{C}$ with $\mathrm{Re}(s) > d$, we have*

$$(1-q^{-s})\zeta^{\mathrm{ask}}_{A(X_U)/\mathfrak{O}}(s) = 1+(1-q^{-1})^{-1} \int\limits_{(\mathfrak{O}V)^\times \times \mathfrak{P}} |z|^{s-n+r-1} \prod_{i=1}^{r} \frac{\|I_{i-1}(x)\|}{\|I_i(x) \cup zI_{i-1}(x)\|} \, \mathrm{d}\mu(x,z).$$

Here and in the following, for an ideal I of $R[X_V]$ and $x \in SV$, we write $I(x)$ for the ideal of S generated by all $f(x)$ as $f(X_V)$ ranges over I. In the context of Proposition 2.2, by Lemma 2.1, we have $I_i(x) = \mathfrak{I}_i(C(x); S)$.

In our applications of Proposition 2.2, the ring R is of the form $R = \mathbf{Z}[1/N]$. In that case, r is simply the rank of $A(X_U)$ over $\mathbf{Q}(X_U)$; in particular, r does not depend on $\mathfrak{O}$.

It is a folklore result in $\mathfrak{P}$-adic integration that zeta functions given by $\mathfrak{P}$-adic integrals defined in terms of monomial ideals are *uniform* in the sense that for some rational function $W(X,T)$, these integrals are of the form $W(q, q^{-s})$ as $\mathfrak{O}$ ranges over (suitable) compact DVRs. The following makes this precise for ask zeta functions.

Proposition 2.3. *Let the notation be as in Proposition 2.2. Suppose that each of $I_1, \ldots, I_r$ is a monomial ideal, say $I_k = \langle X_V^a : a \in A_k \rangle$ for a finite set $A_k \subset \mathbf{N}_0 V$. Then there exists a rational function $W(X,T) \in \mathbf{Q}(X,T)$ (explicitly expressible in terms of n, r, and $A_1, \ldots, A_k$) such that for all compact* DVR*s $\mathfrak{O}$ endowed with an R-algebra structure, we have $\mathsf{Z}^{\mathrm{ask}}_{A(X_U)/\mathfrak{O}}(T) = W(q,T)$.*

Proof. Apply [12, Prop. 3.9] to the affine version [13, Eqn (4.6)] of the integral in Proposition 2.2. ♦

3 (Hyper)graphs and their ask zeta functions

3.1 Hypergraph basics

Two hypergraphs $\mathsf{H} = (V, E, \imath)$ and $\mathsf{H}' = (V', E', \imath')$ are **isomorphic** if there exist bijections $V \xrightarrow{\psi} V'$ and $E \xrightarrow{\phi} E'$ such that for all $v \in V$ and $e \in E$, we have $v \,\imath\, e$ if and only if $v^{\phi} \,\imath'\, e^{\psi}$.

Incidence matrices. Let H have m hyperedges and n vertices. Write $E = \{e_1, \dots, e_m\}$ and $V = \{v_1, \dots, v_n\}$. Equivalently, we endow E and V with (arbitrary) total orders $\preccurlyeq$ and $\sqsubseteq$, respectively, given by $e_1 \preccurlyeq \cdots \preccurlyeq e_m$ and $v_1 \sqsubseteq \cdots \sqsubseteq v_n$. Having made these choices, the associated **incidence matrix** of H is the $(0,1)$-matrix $A_{\mathsf{H}} \in \mathrm{M}_{n\times m}(\mathbf{Z})$ given by $(A_{\mathsf{H}})_{ij} = [v_i \,\imath\, e_j]$ (Iverson bracket notation, see §1.12).

Disjoint unions. Given hypergraphs $\mathsf{H} = (V, E, \imath)$ and $\mathsf{H}' = (V', E', \imath')$, their disjoint union is $\mathsf{H} \oplus \mathsf{H}' = (V \sqcup V', E \sqcup E', \imath \sqcup \imath')$. Given total orders on V and V' (resp. E and E'), we obtain a total order on $V \sqcup V'$ (resp. $E \sqcup E'$) in which the elements of V (resp. E) precede those of V' (resp. E'). With respect to these orders, we then have $\mathsf{A}_{\mathsf{H}\oplus\mathsf{H}'} = \left[\begin{smallmatrix} \mathsf{A}_{\mathsf{H}} & 0 \\ 0 & \mathsf{A}_{\mathsf{H}'} \end{smallmatrix}\right]$.

Subhypergraphs. Let $\mathsf{H} = (V, E, \imath)$ be a hypergraph. A **subhypergraph** of H is a hypergraph $\mathsf{H}' = (V', E', \imath')$ with $V \subset V'$, $E \subset E'$, and $\imath' \subset \imath$. Given subsets $V' \subset V$ and $E' \subset E$, the associated **induced subhypergraph** of H is

$$\mathsf{H}[V' \mid E'] = (V', E', \imath \cap (V' \times E')).$$

Subhypergraphs of (incidence hypergraphs of) graphs will play an important role throughout this article; see §3.3.

Order the vertices and hyperedges of H as above to define the incidence matrix A_{H}. Then the incidence matrix of $\mathsf{H}[V' \mid E']$ relative to the induced total orders on V' and E' is the submatrix of A_{H} obtained by selecting rows from V' and columns from E', respectively.

3.2 Ask zeta functions associated with hypergraphs: $\zeta^{\mathrm{ask}}_{\mathsf{A}_{\mathsf{H}}/\mathfrak{O}}$

Let $\mathsf{H} = (V, E, \imath)$ be a hypergraph with m hyperedges and n vertices. Write $E = \{e_1, \dots, e_m\}$ and $V = \{v_1, \dots, v_n\}$. Let $F = \mathcal{F}(\mathsf{H}) = \{(v, e) \in V \times E : v \,\imath\, e\}$ be the set of **flags** of H. Let $\mathsf{A}_{\mathsf{H}} = \mathsf{A}_{\mathsf{H}}(X_F) \in \mathrm{M}_{n\times m}(\mathbf{Z}[X_F])$ be the matrix of linear forms with

$$\mathsf{A}_{\mathsf{H}}(X_F)_{ij} = \begin{cases} X_{(v_i, e_j)}, & \text{if } v_i \,\imath\, e_j, \\ 0, & \text{otherwise.} \end{cases}$$

Up to equivalence (see §2), $\mathsf{A}_{\mathsf{H}}(X_F)$ only depends on H and not on the chosen total orders. Note that $\mathsf{A}_{\mathsf{H}}(X_F)$ is obtained from the incidence matrix A_{H} of H (w.r.t. the given total orders on V and E) by replacing nonzero entries by distinct variables. We refer to $\mathsf{A}_{\mathsf{H}}(X_F)$ as the **linearised incidence matrix** of H.

Remark 3.1. The zeta function we denote by $\zeta^{\mathrm{ask}}_{\mathsf{A_H}/\mathfrak{O}}(s)$ here coincides with $\mathsf{Z}^{\mathrm{ask}}_{\eta^{\mathfrak{O}}}(q^{-s})$ from [18, §3.2], where η denotes the *incidence representation* of H.

Write $f = |F|$. By ordering F lexicographically relative to the chosen total orders on V and E, we may identify F and $\{1, \dots, f\}$. Let $\mathsf{C_H} = \mathsf{C_H}(X_V) \in \mathrm{M}_{f\times m}(\mathbf{Z}[X_V])$ be the matrix such that for $(v, e) \in F$ and $j \in \{1, \dots m\}$, we have

$$\mathsf{C_H}(X_V)_{(v,e)j} = \begin{cases} X_v, & \text{if } e = e_j, \\ 0, & \text{otherwise.} \end{cases}$$

It is easy to see (cf. [18, §3.2]) that $\mathsf{C_H}$ is a $\circ$-dual of $\mathsf{A_H}(X_F)$. Let $\mathfrak{I}_k\mathsf{H} = \mathfrak{I}_k(\mathsf{C_H}) \subset \mathbf{Z}[X_V]$ denote the ideal of $\mathbf{Z}[X_V]$ generated by the $k \times k$ minors of $\mathsf{C_H}$. In contrast to $\mathsf{C_H}$, the ideal $\mathfrak{I}_k\mathsf{H}$ only depends on H and not on the arbitrary choices of total orders on V, E, and F used to define $\mathsf{C_H}$. Write $r_{\mathsf{H}} = \mathrm{rk}_{\mathbf{Q}(X_V)}(\mathsf{C_H})$; we will derive a simple description of this number in Proposition 7.1 below. For $W \subset \mathfrak{O}V \times \mathfrak{O}$, we define

$$\int\limits_W \mathsf{H}(s) := \int\limits_W |z|^{s-n+r_{\mathsf{H}}-1} \prod_{i=1}^{r_{\mathsf{H}}} \frac{\|\mathfrak{I}_{i-1}\mathsf{H}(x)\|}{\|\mathfrak{I}_i\mathsf{H}(x) \cup z\mathfrak{I}_{i-1}\mathsf{H}(x)\|}\, \mathrm{d}\mu(x, z), \tag{3.1}$$

where μ denotes the normalised Haar measure on $\mathfrak{O}V \times \mathfrak{O}$. Proposition 2.2 (with $R = \mathbf{Z}$) then yields the following.

Proposition 3.2. *For each compact* DVR $\mathfrak{O}$ *and* $s \in \mathbf{C}$ *with* $\mathrm{Re}(s) > n$*, we have*

$$(1 - q^{-s})\zeta^{\mathrm{ask}}_{\mathsf{A_H}/\mathfrak{O}}(s) = 1 + (1 - q^{-1})^{-1} \int\limits_{(\mathfrak{O}V)^\times \times \mathfrak{P}} \mathsf{H}(s). \qquad \blacklozenge$$

3.3 Graph basics

By a **graph**, we mean a pair $\Gamma = (V, E)$ where V is a finite set and E is a set of subsets of V, each of which has cardinality 1 or 2. As usual, we refer to the elements of V and E as the **vertices** and **edges** of Γ, respectively. We explicitly allow loops (i.e. edges e with $|e| = 1$) but no parallel edges. Let $\sim\, = \,\sim_\Gamma \subset V \times V$ be the (symmetric) adjacency relation of Γ. Hence, $v \sim v'$ if and only if $\{v, v'\} \in E$. By a **subgraph** of Γ, we mean a graph $\Gamma' = (V', E')$ with $V' \subset V$ and $E' \subset E$.

Hypergraphs from graphs. Let $\imath = \imath_\Gamma \subset V \times E$ be the incidence relation of Γ. (Hence, $v \,\imath\, e$ if and only if $v \in e$.) Every graph gives rise to two hypergraphs that will be of interest to us: the **incidence hypergraph** $\mathscr{Inc}(\Gamma) = (V, E, \imath_\Gamma)$ and the **adjacency hypergraph** $\mathscr{Adj}(\Gamma) = (V, V, \sim_\Gamma)$. The former of these simply amounts to viewing a graph Γ as a hypergraph whose hyperedges are the edges of Γ with the evident incidence relation.

Disjoint unions and joins. Given graphs Γ and Γ', we let $\Gamma \oplus \Gamma'$ denote their disjoint union. We have $\mathscr{Inc}(\Gamma \oplus \Gamma') = \mathscr{Inc}(\Gamma) \oplus \mathscr{Inc}(\Gamma')$ and $\mathscr{Adj}(\Gamma \oplus \Gamma') = \mathscr{Adj}(\Gamma) \oplus \mathscr{Adj}(\Gamma')$. The **join** $\Gamma \vee \Gamma'$ of Γ and Γ' is obtained from $\Gamma \oplus \Gamma'$ by adding an edge connecting each vertex of Γ to each vertex of Γ'.

Adjacency and incidence matrices. Let Γ have n vertices, say $V = \{v_1, \dots, v_n\}$. As usual, the **adjacency matrix** $A_\Gamma \in \mathrm{M}_n(\mathbf{Z})$ of Γ (relative to the chosen total order on V) is the $(0,1)$-matrix given by $(A_\Gamma)_{ij} = [v_i \sim v_j]$. Note that, using the same total order on V throughout, the adjacency matrix A_Γ of Γ coincides with the incidence matrix $A_{\mathscr{A}dj(\Gamma)}$ of the adjacency hypergraph of Γ.

3.4 Two ask zeta functions associated with graphs: $\zeta^{\mathrm{ask}}_{\mathsf{A}^+_\Gamma/\mathfrak{O}}$ and $\zeta^{\mathrm{ask}}_{\mathsf{A}^-_\Gamma/\mathfrak{O}}$

Let $\Gamma = (V, E)$ be a graph with m edges and n vertices. As in the previous section, we write $V = \{v_1, \dots, v_n\}$ which reflects a choice of a total order $\preccurlyeq$ on V with $v_1 \preccurlyeq \dots \preccurlyeq v_n$. Each edge $e \in E$ is of the form $\{v_i, v_j\}$ with $i \leqslant j$ and $v_i \sim v_j$. We obtain a total order on E, which we again denote by $\preccurlyeq$, by mapping e to (i, j) and by ordering the resulting pairs of numbers lexicographically. Let $e_1 \preccurlyeq \dots \preccurlyeq e_m$ be the distinct edges of Γ. Using this order, we identify E and $\{1, \dots, m\}$.

Following (and, in fact, slightly generalising) [18], we now define matrices $\mathsf{A}^+_\Gamma(X_E)$ and $\mathsf{A}^-_\Gamma(X_E)$ in $\mathrm{M}_{n\times n}(\mathbf{Z}[X_E])$. Namely, $\mathsf{A}^\pm_\Gamma$ is the matrix given by the following conditions:

- For $1 \leqslant i \leqslant j \leqslant n$, we have
$$\mathsf{A}^\pm_\Gamma(X_E)_{ij} = \begin{cases} X_e, & \text{if } e := \{v_i, v_j\} \in E, \\ 0, & \text{otherwise.} \end{cases}$$
- For $1 \leqslant i < j \leqslant n$, we have $\mathsf{A}^\pm_\Gamma(X_E)_{ij} = \pm\mathsf{A}^\pm_\Gamma(X_E)_{ji}$.

We refer to A^+_Γ and A^-_Γ as the **linearised adjacency matrices** of Γ. Note that A^+_Γ is symmetric and if Γ is loopless, then A^-_Γ is antisymmetric. The matrix $\mathsf{A}^-_\Gamma + (\mathsf{A}^-_\Gamma)^\top$ is diagonal and the (i,i)-entry of $\mathsf{A}^\pm_\Gamma$ is $X_{\{v_i\}}$ if $v_i \sim v_i$ in Γ and zero otherwise.

Remark 3.3. The zeta function we denote by $\zeta^{\mathrm{ask}}_{\mathsf{A}^\pm_\Gamma/\mathfrak{O}}(s)$ here coincides with $\mathsf{Z}^{\mathrm{ask}}_{\gamma^{\mathfrak{O}}_\pm}(q^{-s})$ from [18], where $\gamma_\pm$ denotes the *(positive or negative) adjacency representation* of Γ [18, §3.2]. We note that Γ was assumed to be loopless in the definition of γ_- in [18].

Let $\mathsf{C}^\pm_\Gamma = \mathsf{C}^\pm_\Gamma(X_V) \in \mathrm{M}_{m\times n}(\mathbf{Z}[X_V])$ be the matrix defined as follows: given an edge $e = \{v_i, v_j\}$ with $i \leqslant j$ and $k \in \{1, \dots, n\}$, we define
$$(\mathsf{C}^\pm_\Gamma)_{ek} = \begin{cases} X_i, & \text{if } k = j, \\ \pm X_j, & \text{if } k = i \text{ and } i \neq j, \\ 0, & \text{otherwise.} \end{cases}$$
It is easy to see (cf. [18, §3.3]) that $\mathsf{C}^\pm_\Gamma(X_V)$ is a $\circ$-dual of $\mathsf{A}^\pm_\Gamma(X_F)$.

Let $\mathfrak{I}^\pm_k\Gamma = \mathfrak{I}_k(\mathsf{C}^\pm_\Gamma)$, an ideal of $\mathbf{Z}[X_V]$. We also write $\frac{1}{2}\mathfrak{I}^+_k\Gamma$ for the ideal of $(\mathbf{Z}[\frac{1}{2}])[X_V]$ generated by $\mathfrak{I}^+_k\Gamma$. As for hypergraphs, all of these ideals only depend on Γ and the chosen "type", $+$ or $-$, not on the total order on the vertices of Γ used to define $\mathsf{C}^\pm_\Gamma$.

Write $r_\Gamma[\pm] = \mathrm{rk}_{\mathbf{Q}(X_V)}(\mathsf{C}_\Gamma^\pm)$; we will obtain graph-theoretic interpretations of these numbers in Proposition 7.2 and Proposition 7.5. For $W \subset \mathfrak{O}V \times \mathfrak{O}$, we define

$$\int\limits_W \Gamma^\pm(s) := \int\limits_W |z|^{s-n+r_\Gamma[\pm]-1} \prod_{i=1}^{r_\Gamma[\pm]} \frac{\|\mathfrak{I}_{i-1}^\pm \Gamma(x)\|}{\|\mathfrak{I}_i^\pm \Gamma(x) \cup z\mathfrak{I}_{i-1}^\pm \Gamma(x)\|} \,\mathrm{d}\mu(x,z), \tag{3.2}$$

where μ is the normalised Haar measure on $\mathfrak{O}V \times \mathfrak{O}$. Proposition 2.2 then yields the following.

Proposition 3.4. *For each compact* DVR $\mathfrak{O}$ *and* $s \in \mathbf{C}$ *with* $\mathrm{Re}(s) > n$*, we have*

$$(1-q^{-s})\zeta^{\mathrm{ask}}_{\mathsf{A}_\Gamma^\pm/\mathfrak{O}}(s) = 1 + (1-q^{-1})^{-1} \int\limits_{(\mathfrak{O}V)^\times \times \mathfrak{P}} \Gamma^\pm(s). \qquad \blacklozenge$$

Remark 3.5. If $\mathfrak{O}$ has odd residue characteristic in the $+$-case of Proposition 3.4, then we may replace each $\mathfrak{I}_i^+\Gamma$ by $\frac{1}{2}\mathfrak{I}_i^+\Gamma$ in the definition of $\int_W \Gamma^+(s)$.

4 Selectors of hypergraphs

Let H be a hypergraph. We derive an explicit combinatorial parameterisation of the minors of C_H (see §3.2) in terms of partial functions that we call *selectors*. Apart from immediately providing us with a new proof of Theorem 1.3(i), the results of this section will play a key role in our proof of Theorem B in §7.

Partial functions. Given a set U, we let $U_\perp$ denote the set obtained from U by adjoining an additional element $\perp$. We write $\phi\colon U \dashrightarrow V$ and $U \overset{\phi}{\dashrightarrow} V$ to indicate a partial function ϕ from U to V. The **domain of definition** $\mathcal{D}(\phi)$ of ϕ consists of those $u \in U$ for which u^ϕ is defined. We tacitly identify partial functions $\phi\colon U \dashrightarrow V$ and those total functions $U_\perp \to V_\perp$ which send $\perp$ to $\perp$. For $Y \subset V_\perp$, write $Y^{\phi^*} = \{u \in U_\perp : u^\alpha \in Y\}$. In particular, $\mathcal{D}(\phi) = \{u \in U : u^\phi \neq \perp\} = V^{\phi^*}$. We write $\deg(\phi) = |\mathcal{D}(\phi)|$ for the **degree** of ϕ. Given $U \overset{\phi}{\dashrightarrow} V$, let

$$\mathrm{mon}(\phi) = \prod_{u \in \mathcal{D}(\phi)} X_{u^\phi} \in \mathbf{Z}[X_V].$$

Note that $\deg(\phi)$ is the (total) degree of $\mathrm{mon}(\phi)$ as a monomial in $\mathbf{Z}[X_V]$.

We will often find the need to modify or extend partial functions. Let $U \overset{\phi}{\dashrightarrow} V$ be a partial function. Let $u_1, \ldots, u_r \in U$ be distinct elements and let $v_1, \ldots, v_r \in V$ be arbitrary. We define

$$\phi[u_1 \leftarrow v_1, \ldots, u_r \leftarrow v_r]$$

to be the partial function $U \dashrightarrow V$ given by

$$u \mapsto \begin{cases} v_i, & \text{if } u = u_i, \\ u^\phi, & \text{otherwise.} \end{cases}$$

Let $U' \subset U$. We identify partial functions $U' \dashrightarrow V$ and those partial functions $U \dashrightarrow V$ that are undefined outside of U'. Given $U \overset{\phi}{\dashrightarrow} V$, we let $\phi \restriction U'$ denote the partial function

$$u \mapsto \begin{cases} u^\phi, & \text{if } u \in U', \\ \perp, & \text{otherwise.} \end{cases}$$

Selectors. Let $\mathsf{H} = (V, E, \imath)$ be a hypergraph. By a **(vertex) selector** of H, we mean a partial function $E \overset{\phi}{\dashrightarrow} V$ such that $e^\phi \ \imath \ e$ for all $e \in \mathcal{D}(\phi)$. In other words, a selector ϕ of H consists of a subset $E' = \mathcal{D}(\phi)$ of E together with a choice of a vertex e^ϕ incident with $e \in E'$. Let $\mathrm{Sel}(\mathsf{H})$ be the set of all selectors of H and $\mathrm{Sel}_k(\mathsf{H}) = \{\phi \in \mathrm{Sel}(\mathsf{H}) : \deg(\phi) = k\}$.

Minors. Using Propositions 2.3 and 3.2, the following description of the minors of $\mathsf{C_H}$ provides a new proof of Theorem 1.3(i), quite different from the one in [18, §4.4].

Proposition 4.1. *We have* $\mathfrak{I}_k\mathsf{H} = \Big\langle \mathrm{mon}(\phi) : \phi \in \mathrm{Sel}_k(\mathsf{H}) \Big\rangle$. *More precisely, the nonzero minors of* $\mathsf{C_H}$ *are precisely of the form* $\pm\,\mathrm{mon}(\phi)$ *for* $\phi \in \mathrm{Sel}(\mathsf{H})$.

Proof. Let H have m hyperedges and n vertices. Write $E = \{e_1, \dots, e_m\}$ and $V = \{v_1, \dots, v_n\}$. Define and order $F = \mathcal{F}(\mathsf{H})$ as in §3.2. Given $F' \subset F$ and $E' \subset E$ with $|F'| = |E'| = k$, let $\mathsf{C_H}[E' \mid F']$ be the submatrix of $\mathsf{C_H}$ with rows indexed by F' and columns indexed by E'. Write $m_{\mathsf{H}}[E' \mid F'] = \det(\mathsf{C_H}[E' \mid F'])$ for the associated minor.

We first show that given F' and E' as above, either $m_{\mathsf{H}}[E' \mid F'] = 0$ or $m_{\mathsf{H}}[E' \mid F'] = \pm\,\mathrm{mon}(\phi)$ for some $\phi \in \mathrm{Sel}_k(\mathsf{H})$. Let $(u_1, f_1), \dots, (u_k, f_k)$ be the distinct elements of F'. If $f_i = f_j$ for $i \neq j$, then the rows of $\mathsf{C_H}[E' \mid F']$ indexed by (u_i, f_i) and (u_j, f_j) are linearly dependent over $\mathbf{Q}(X_V)$ whence $m_{\mathsf{H}}[E' \mid F'] = 0$. We may thus assume that $E'' := \{f_1, \dots, f_k\}$ has cardinality k. Next, if $f_i \notin E'$ for some i, then the row of $\mathsf{C_H}[E' \mid F']$ indexed by (u_i, f_i) is zero so that again $m_{\mathsf{H}}[E' \mid F'] = 0$. Therefore, we may assume that $E' = E''$. In this case, we clearly have $m_{\mathsf{H}}[E' \mid F'] = \pm \prod_{i=1}^k X_{u_i}$. Define a selector $E \overset{\phi}{\dashrightarrow} V$ via $\mathcal{D}(\phi) = E'$ and $f_i^\phi = u_i$ for $i = 1, \dots, k$. Then $m_{\mathsf{H}}[E' \mid F'] = \pm\,\mathrm{mon}(\phi)$.

Conversely, given $\phi \in \mathrm{Sel}(\mathsf{H})$, let $E' = \mathcal{D}(\phi)$ and $F' = \{(e^\phi, e) : e \in E'\}$ so that $m_{\mathsf{H}}[E' \mid F'] = \pm\,\mathrm{mon}(\phi)$. ♦

Our proof of Proposition 4.1 will act as a template for the much more involved case of the matrices $\mathsf{C}_\Gamma^\pm$ in §6.

Corollary 4.2. *Let* $\mathsf{H} = (V, E, \imath)$ *be a hypergraph. Then*

$$\mathrm{rk}_{\mathbf{Q}(X_V)}(\mathsf{C_H}) = \max(k \geqslant 0 : \mathrm{Sel}_k(\mathsf{H}) \neq \emptyset) = \max(\deg(\phi) : \phi \in \mathrm{Sel}(\mathsf{H})). \quad ♦$$

New proof of Theorem 1.3(i). Combine Proposition 2.3, Proposition 3.2, and Proposition 4.1. ♦

5 Towards minors: paths and cycles in hypergraphs

Our next goal, to be achieved in §6, is to provide a combinatorial parameterisation of the minors of the matrices $\mathsf{C}_\Gamma^{\pm}$ associated with a graph $\Gamma = (V, E)$ in §3.4. The columns and rows of $\mathsf{C}_\Gamma^{\pm}$ correspond to elements of V and E, respectively. Given subsets $V' \subset V$ and $E' \subset E$ with $|V'| = |E'| = k$, we therefore obtain an associated $k \times k$ minor $m_\Gamma^{\pm}[V' \mid E']$ of $\mathsf{C}_\Gamma^{\pm}$. It turns out that these minors can be conveniently studied in terms of the subhypergraphs $\mathsf{H}[V' \mid E']$ of the incidence hypergraph $\mathsf{H} = \mathscr{Inc}(\Gamma)$ of Γ.

5.1 Degeneracy and connectivity of hypergraphs

Let $\mathsf{H} = (V, E, \imath)$ be a hypergraph. Two hyperedges $e, e' \in E$ are **parallel** if $\|e\| = \|e'\|$. If $\|e\| \neq \|e'\|$ whenever $e \neq e'$ for $e, e' \in E$, then H is **simple**. A **loop** of H is a hyperedge $e \in E$ with $\#\|e\| = 1$. Following [4, §1.1], we call $\mathrm{iso}(\mathsf{H}) = \{v \in V : v \not\imath\, e \text{ for all } e \in E\}$ and $\mathrm{emp}(\mathsf{H}) = \{e \in E : \|e\| = \emptyset\}$ the sets of **isolated** vertices and **empty** hyperedges of H, respectively. We call H **nondegenerate** if $\mathrm{iso}(\mathsf{H}) = \mathrm{emp}(\mathsf{H}) = \emptyset$.

The following notion of connectivity of hypergraphs is adapted from [4, §1.2]. Given vertices $u, v \in V$, a **walk** of length $r \geqslant 0$ from u to v is a sequence

$$u = u_1,\, e_1,\, u_2,\, e_2,\, \ldots,\, u_r,\, e_r,\, u_{r+1} = v$$

with $u_1, \ldots, u_{r+1} \in V$ and $e_1, \ldots, e_r \in E$ such that always $u_i \,\imath\, e_i$ and $u_{i+1} \,\imath\, e_i$.

We say that vertices $u, v \in V$ of H are **connected** if there exists a walk from u to v. This defines an equivalence relation on V. Let $V_1, \ldots, V_c$ be the distinct equivalence classes. Write $E_j = \{e \in E : \|e\| \cap V_j \neq \emptyset\}$ and define $\mathsf{H}_j = \mathsf{H}[V_j \mid E_j]$ to be the associated induced subhypergraph. Let $\mathsf{H}_0 = (\emptyset, \mathrm{emp}(\mathsf{H}), \emptyset)$. It is then easy to see that $\mathsf{H} = \mathsf{H}_0 \oplus \cdots \oplus \mathsf{H}_c$. We refer to the subhypergraphs $\mathsf{H}_1, \ldots, \mathsf{H}_c$ as the **connected components** of H. We say that H is **connected** if $\mathrm{emp}(\mathsf{H}) = \emptyset$ and $c = 1$. (Note that using this definition, every hypergraph without vertices is disconnected.) If Γ is a graph, then $\mathscr{Inc}(\Gamma)$ is connected if and only if Γ is connected in the usual graph-theoretic sense.

Lemma 5.1. *A hypergraph $\mathsf{H} = (V, E, \imath)$ is connected if and only if $V \neq \emptyset$, $\mathrm{emp}(\mathsf{H}) = \emptyset$, and whenever $\mathsf{H} = \mathsf{H}_1 \oplus \mathsf{H}_2$ for hypergraphs H_1 and H_2, one of the summands is $(\emptyset, \emptyset, \emptyset)$.*

Proof. Suppose that H is connected. As connectivity yields a (single) equivalence class on V, we have $V \neq \emptyset$. Suppose that $\mathsf{H} = \mathsf{H}_1 \oplus \mathsf{H}_2$ for $\mathsf{H}_j = (V_j, E_j, \imath_j)$. If $v_j \in V_j$ for $j = 1, 2$, then v_1 and v_2 cannot be connected in H. Since H is connected, we may assume that, without loss of generality, $V_1 = \emptyset$. As $\mathrm{emp}(\mathsf{H}) = \emptyset$, we then obtain $E_1 = \emptyset = \imath_1$.

Conversely, suppose that H satisfies the conditions stated. Write $\mathsf{H} = \mathsf{H}_0 \oplus \cdots \oplus \mathsf{H}_c$ as in the paragraph preceding this lemma. Since $V \neq \emptyset$, we have $c \geqslant 1$ and since $\mathrm{emp}(\mathsf{H}) = \emptyset$, we have $\mathsf{H}_0 = (\emptyset, \emptyset, \emptyset)$. Hence, $\mathsf{H} = \mathsf{H}_1 \oplus \cdots \oplus \mathsf{H}_c$ whence $c = 1$ by assumption. ♦

The following lemma simply asserts that the determinant of a block diagonal square matrix (with entries in some ring) can only be nonzero if all diagonal blocks are squares.

Lemma 5.2. *Let $\mathsf{H} = \mathsf{H}_1 \oplus \cdots \oplus \mathsf{H}_r$, where each $\mathsf{H}_j = (V_j, E_j, \imath_j)$ is a hypergraph. Write $m_j = |E_j|$ and $n_j = |V_j|$. Suppose that $\sum_{j=1}^{r} m_j = \sum_{j=1}^{r} n_j$.*

(i) If $m_j \neq n_j$ *for some* j*, then* $\det(\mathsf{A}_\mathsf{H}) = 0$.

(ii) If $m_j = n_j$ *for all* j*, then* $\det(\mathsf{A}_\mathsf{H}) = \pm \prod_{j=1}^{r} \det(\mathsf{A}_{\mathsf{H}_j})$.

Proof. Only (i) merits a proof. If $m_j \neq n_j$ for some j and $\sum_{j=1}^{r} m_j = \sum_{j=1}^{r} n_j$, then $n_j > m_j$ for some j. The rows of $\mathsf{A}_{\mathsf{H}_j} \in \mathrm{M}_{n_j \times m_j}(\mathbf{Z}[X_{V_j}])$ are then linearly dependent over $\mathbf{Q}(X_{V_j})$ whence $\det(\mathsf{A}_\mathsf{H}) = 0$. ♦

5.2 Hypergraph decompositions of graphs and nonzero minors

Let $\Gamma = (V, E)$ be a graph with associated incidence hypergraph $\mathsf{H} = \mathscr{Inc}(\Gamma) = (V, E, \imath)$. The minors of $\mathsf{C}_\Gamma^{\pm}$ are parameterised by pairs (V', E') with $V' \subset V$, $E' \subset E$, and $|V'| = |E'|$. While such a pair (V', E') may not be a subgraph of Γ, we may always consider the associated induced subhypergraph $\mathsf{H}' = \mathsf{H}[V' \mid E']$ of H as in §3.1.

Let Γ have m edges and n vertices. Order and index the vertices and edges of Γ as in §3.4. Given $V' \subset V$ and $E' \subset E$, we let $\mathsf{C}_\Gamma^{\pm}[V' \mid E']$ denote the submatrix of $\mathsf{C}_\Gamma^{\pm}$ obtained by selecting the rows indexed by elements of E' and the columns indexed by elements of V'. Henceforth, we assume that $|V'| = |E'| = k$. We write $m_\Gamma^{\pm}[V' \mid E'] = \det(\mathsf{C}_\Gamma^{\pm}[V' \mid E'])$ for the minor of $\mathsf{C}_\Gamma^{\pm}$ corresponding to (V', E').

Lemma 5.3. *If the hypergraph* $\mathsf{H}[V' \mid E']$ *is degenerate, then* $m_\Gamma^{\pm}[V' \mid E'] = 0$.

Proof. An isolated vertex (resp. empty hyperedge) of $\mathsf{H}[V' \mid E']$ gives rise to a zero column (resp. zero row) of $\mathsf{C}_\Gamma^{\pm}[V' \mid E']$. ♦

Let $\mathsf{H}' = \mathsf{H}[V' \mid E']$ be nondegenerate. Let $\mathsf{H}' = \mathsf{H}'_1 \oplus \cdots \oplus \mathsf{H}'_c$ be its decomposition into connected components. Write $\mathsf{H}'_j = \mathsf{H}[V'_j \mid E'_j]$, where $V' = V'_1 \sqcup \cdots \sqcup V'_c$ and $E' = E'_1 \sqcup \cdots \sqcup E'_c$. We call a hypergraph **square** if it has as many hyperedges as vertices.

Lemma 5.4.

(i) If some H'_j *is nonsquare, then* $m_\Gamma^{\pm}[V' \mid E'] = 0$.

(ii) If each H'_j *is square, then* $m_\Gamma^{\pm}[V' \mid E'] = \pm \prod_{j=1}^{c} m_\Gamma^{\pm}[V'_j \mid E'_j]$.

Proof. By changing our total order on V (resp. E) if needed, we may assume that each element of V'_j (resp. E'_j) precedes every element of V'_{j+1} (resp. E'_{j+1}). The matrix $\mathsf{C}_\Gamma^{\pm}[V' \mid E']$ is then a block diagonal matrix with blocks $\mathsf{C}_\Gamma^{\pm}[V'_j \mid E'_j]$. Both claims then follow from Lemma 5.2. Indeed, every $a \times b$ matrix (with entries in a ring) is obtained by specialising $\mathsf{A}_{\mathsf{H}_{a,b}}$, where, following [18, §3.1], $\mathsf{H}_{a,b}$ is the block hypergraph with a vertices and b hyperedges such that every vertex is incident with every hyperedge. ♦

Remark 5.5. Even though Γ is assumed to be simple as a graph—equivalently, $\mathscr{Inc}(\Gamma)$ is assumed simple as a hypergraph—H' need not be simple. In particular, suppose that $e = \{u, v\}$ and $f = \{u, w\}$ with $v \neq w$ are edges of Γ that both belong to E'. Further suppose that $u \in V'$ but $v, w \notin V'$. Then e and f are distinct parallel loops of $\mathsf{H}[V' \mid E']$.

To summarise: in studying the minors $m_\Gamma^{\pm}[V' \mid E']$, we obtained a reduction to the case that H' is nondegenerate, connected, and square.

5.3 Unicyclic graphs and related hypergraphs

Following [10], a graph is **unicyclic** if it is connected and contains a unique cycle. Equivalently, a graph is unicyclic if and only if it is connected and contains as many vertices as edges. Unicyclic graphs are precisely those graphs obtained from a tree by adding an edge connecting two previously non-adjacent (not necessarily distinct!) vertices.

In the preceding subsection, we reduced the study of the minors $m_\Gamma^\pm[V' \mid E']$ to the case when $\mathsf{H}[V' \mid E']$ is connected and nondegenerate, where $\mathsf{H} = \mathscr{Inc}(\Gamma)$. The following outlines the highly restricted possible shapes of the subhypergraphs $\mathsf{H}[V' \mid E']$.

Lemma 5.6 ("Unicyclicity lemma")**.** *Let $\Gamma = (V, E)$ be a graph and $\mathsf{H} = \mathscr{Inc}(\Gamma)$. Let $V' \subset V$ and $E' \subset E$ have the same cardinality. Let $\mathsf{H}' = \mathsf{H}[V' \mid E']$ be connected and nondegenerate. Then precisely one of the following conditions is satisfied.*

(U1) (V', E') is a unicyclic loopless subgraph of Γ and H' is its incidence hypergraph.

(U2) H' contains a unique loop $e_\circ$ and $(V', E' \setminus \{e_\circ\})$ is a subtree of Γ.

Proof. We write $k = |V'| = |E'|$. As H' is nondegenerate and Γ is a graph, every hyperedge of H' contains either one or two vertices. Let $E'' \subset E'$ consist of those $e \in E'$ with $\#\|e\|_{\mathsf{H}'} = 2$. As loops are irrelevant for connectivity, the hypergraph $\mathsf{H}'' = \mathsf{H}[V' \mid E'']$ is still connected. Note that this hypergraph is the incidence hypergraph of the loopless graph $\Gamma'' = (V', E'')$. It follows that Γ'' is connected (as a graph) with k vertices. Therefore, Γ'' contains at least $k-1$ edges. On the other hand, since $E'' \subset E'$, the graph Γ'' contains at most k edges. This leaves us with two cases.

(i) If $E'' = E'$ (i.e. Γ'' has k edges), then H' is the incidence hypergraph of the unicyclic loopless graph (V', E').

(ii) Otherwise, H' contains a unique loop $e_\circ$, we have $E' = E'' \cup \{e_\circ\}$, and $\Gamma'' = (V', E' \setminus \{e_\circ\})$ is a tree. The edge $e_\circ$ may or may not be a loop of Γ; cf. Remark 5.5. ♦

6 Animations of graphs

Proposition 4.1 not only shows that each ideal $\mathfrak{I}_k\mathsf{H}$ of minors attached to a hypergraph H is monomial. It also provides explicit (monomial) generators parameterised by combinatorial gadgets, namely selectors. This parameterisation is not faithful: it is easy to produce examples of distinct selectors giving rise to the same minor. In this section, we will derive similar (albeit more delicate) parameterisations of monomial generators of $\mathfrak{I}_k^-\Gamma$ and $\frac{1}{2}\mathfrak{I}_k^+\Gamma$ by means of what we call *animations* of Γ. These parameterisations are not faithful either. In later sections, this will emerge as a useful feature which allows us to manipulate animations by means of combinatorial procedures.

Throughout this section, let $\Gamma = (V, E)$ be a graph with incidence hypergraph $\mathsf{H} = \mathscr{Inc}(\Gamma)$; see §3.3.

6.1 Nilpotency, periodic, and transient points

Let U be a finite set. Let $\mathrm{Par}(U)$ denote the set of all partial functions $U \dashrightarrow U$; see §4. This is a monoid with respect to composition (of functions $U_\perp \to U_\perp$) with zero element given by the nowhere defined function. In particular, we have a natural notion of nilpotency for elements of $\mathrm{Par}(U)$. Namely, $\phi \in \mathrm{Par}(U)$ is **nilpotent** if there exists some $n \geqslant 1$ such that the n-fold composite ϕ^n sends all points of U to $\perp$.

Let $U \overset{\phi}{\dashrightarrow} U$ be given. Borrowing terminology from finite dynamical systems, we call $u \in U$ a **ϕ-periodic** point if $u^{\phi^n} = u$ for some $n \geqslant 1$; otherwise, u is a **ϕ-transient** point. Let $U^{\mathrm{per}} \subset U$ consist of all ϕ-periodic points. Then $U^{\mathrm{per}}_\perp = U^{\mathrm{per}} \sqcup \{\perp\}$ is the set of periodic points of ϕ viewed as a total function $U_\perp \to U_\perp$. Let $U^{\mathrm{tra}} \subset V$ be the set of ϕ-transient points on $U_\perp$. That is,

$$U^{\mathrm{tra}} = \{u \in U : u^{\phi^k} \in U^{\mathrm{per}}_\perp \text{ for some } k \geqslant 1\}.$$

Clearly, ϕ induces a permutation of U^{per}. By the **ϕ-orbits** on U^{per}, we mean the orbits of the infinite cyclic group acting on U^{per} via ϕ. Let $\mathscr{o\ell}(\phi)$ denote the number of ϕ-orbits of odd length > 1. We call ϕ **odd-periodic** if all ϕ-periodic points have odd ϕ-periods.

6.2 Animations

An **animation** of Γ is a partial function $V \overset{\alpha}{\dashrightarrow} V$ such that $v^\alpha \sim v$ for all $v \in V$ with $v^\alpha \neq \perp$. Equivalently, an animation of Γ is a selector of the adjacency hypergraph $\mathscr{Adj}(\Gamma)$ (see §3.3). We write $\mathrm{Ani}(\Gamma)$ for the set of all animations of Γ. Let $\mathrm{Nil}(\Gamma)$ and $\mathrm{Odd}(\Gamma)$ denote the set of nilpotent and odd-periodic animations of Γ, respectively. Let $\mathrm{Fix}(\Gamma)$ denote the set of those animations $\alpha \in \mathrm{Ani}(\Gamma)$ such that every α-periodic point is a fixed point of α. We clearly have

$$\mathrm{Nil}(\Gamma) \subset \mathrm{Fix}(\Gamma) \subset \mathrm{Odd}(\Gamma) \subset \mathrm{Ani}(\Gamma).$$

We write $\mathrm{Nil}_k(\Gamma)$, $\mathrm{Fix}_k(\Gamma)$, $\mathrm{Odd}_k(\Gamma)$, and $\mathrm{Ani}_k(\Gamma)$ for the respective subsets consisting of animations of degree k. The following result, proved over the course of this section, is the key ingredient of all main results of the present article.

Theorem 6.1. *Let Γ be a graph and $k \geqslant 0$. Then:*

(i) $\mathfrak{I}^-_k\Gamma = \left\langle \mathrm{mon}(\alpha) : \alpha \in \mathrm{Fix}_k(\Gamma) \right\rangle$. *More precisely, the nonzero minors of C^-_Γ are precisely of the form $\pm\,\mathrm{mon}(\alpha)$ for $\alpha \in \mathrm{Fix}(\Gamma)$.*

(ii) $\frac{1}{2}\mathfrak{I}^+_k\Gamma = \left\langle \mathrm{mon}(\alpha) : \alpha \in \mathrm{Odd}_k(\Gamma) \right\rangle$. *More precisely, the nonzero minors of C^+_Γ are precisely of the form $\pm 2^{\mathscr{o\ell}(\alpha)}\,\mathrm{mon}(\alpha)$ for $\alpha \in \mathrm{Odd}(\Gamma)$.*

Corollary 6.2. *Let $\Gamma = (V, E)$ be a graph. Then*

$$\begin{aligned}&\mathrm{rk}_{\mathbf{Q}(X_V)}(\mathsf{C}^-_\Gamma) = \max(k \geqslant 0 : \mathrm{Fix}_k(\Gamma) \neq \emptyset) = \max(\deg(\alpha) : \alpha \in \mathrm{Fix}(\Gamma)) \text{ and}\\ &\mathrm{rk}_{\mathbf{Q}(X_V)}(\mathsf{C}^+_\Gamma) = \max(k \geqslant 0 : \mathrm{Odd}_k(\Gamma) \neq \emptyset). = \max(\deg(\alpha) : \alpha \in \mathrm{Odd}(\Gamma)). \qquad \blacklozenge\end{aligned}$$

For group-theoretic applications, the ask zeta functions $W_\Gamma^-(X,T)$ (see Theorem 1.3) are of particular interest when Γ is loopless.

Corollary 6.3. *Let Γ be loopless. Then $\mathfrak{I}_k^-\Gamma = \left\langle \mathrm{mon}(\alpha) : \alpha \in \mathrm{Nil}_k(\Gamma)\right\rangle$. More precisely, the nonzero minors of C_Γ^- are precisely of the form $\pm\,\mathrm{mon}(\alpha)$ for $\alpha \in \mathrm{Nil}(\Gamma)$.*

Proof. If Γ is loopless, then $\mathrm{Nil}(\Gamma) = \mathrm{Fix}(\Gamma)$. Now apply Theorem 6.1(i). ♦

In §6.3 we prove that (suitable) animations give rise to minors of $\mathsf{C}_\Gamma^\pm$, proving "half" of Theorem 6.1. In Sections 6.4–6.5 we establish the other half by showing that all nonzero minors of $\mathsf{C}_\Gamma^\pm$ arise from suitable animations.

6.3 Animations yield minors

We begin by proving half of Theorem 6.1: we show that if $\alpha \in \mathrm{Fix}(\Gamma)$ (resp. $\alpha \in \mathrm{Odd}(\Gamma)$), then $\pm 2^{\mathscr{o\ell}(\alpha)}\,\mathrm{mon}(\alpha)$ is a minor of C_Γ^- (resp. C_Γ^+).

Notation for submatrices and minors of $\mathsf{C}_\Gamma^\pm$. We use the following notation (which builds upon on that from §5.2) throughout this entire section. Recall that the rows and columns of $\mathsf{C}_\Gamma^\pm$ are indexed by edges and vertices of Γ, respectively. As we are only interested in minors up to signs, we are free to order vertices and edges as we see fit. If $v_1, \dots, v_n$ are the distinct vertices of Γ, then, up to multiplication by ± 1, each nonloop edge $\{v_i, v_j\}$ of Γ gives rise to a row $X_{v_i}\mathsf{b}_j \pm X_{v_j}\mathsf{b}_i \in \mathbf{Z}[X_V]^n$ of $\mathsf{C}_\Gamma^\pm$, while each loop $\{v_i\}$ gives rise to a row $X_{v_i}\mathsf{b}_i$. Having ordered V, we order E (e.g. lexicographically as in §3.4) to define $\mathsf{C}_\Gamma^\pm$. Given subsets $V' \subset V$ and $E' \subset E$, we then obtain a submatrix $\mathsf{C}_\Gamma^\pm[V' \mid E']$ of $\mathsf{C}_\Gamma^\pm$ obtained by selecting the columns indexed using the elements of V' (in their chosen order) and the rows indexed using the elements of E' (again in their chosen order). If $|V'| = |E'| = k$, then we write $m_\Gamma^\pm[V' \mid E'] = \det(\mathsf{C}_\Gamma^\pm[V' \mid E'])$ for the corresponding $k \times k$ minor of $\mathsf{C}_\Gamma^\pm$.

Partitioning vertices using an animation. Let $\alpha \in \mathrm{Ani}(\Gamma)$. We describe a canonical partition of the vertices of Γ which will play a crucial role in our subsequent construction of a minor of $\mathsf{C}_\Gamma^\pm$ related to $\mathrm{mon}(\alpha)$.

As in §6.1, let $V^{\mathrm{per}} \subset V$ (resp. $V^{\mathrm{tra}} \subset V$) be the set of α-periodic (resp. α-transient) points in V. Let $V_\perp^{\mathrm{per}} = \bigsqcup_{\lambda\in\Lambda} \Omega_\lambda$ be the decomposition into orbits of the infinite cyclic group acting on $V_\perp^{\mathrm{per}}$ via α. We assume that $0 \in \Lambda$ and $\Omega_0 = \{\perp\}$. We write $\Lambda^+ = \Lambda \setminus \{0\}$ and $N_\lambda = |\Omega_\lambda|$. By definition, $\alpha \in \mathrm{Odd}(\Gamma)$ if and only if each N_λ is odd, in which case $\mathscr{o\ell}(\alpha) = \#\{\lambda \in \Lambda : N_\lambda > 1\}$. Furthermore, $\alpha \in \mathrm{Fix}(\Gamma)$ if and only if $N_\lambda = 1$ for all $\lambda \in \Lambda$. Finally, $\alpha \in \mathrm{Nil}(\Gamma)$ if and only if $\Lambda = \{0\}$.

By repeated application of α, every transient point $v \in V^{\mathrm{tra}}$ is moved into precisely one of the orbits, $\Omega_{\omega(v)}$ say. We let $\delta(v)$ be the least positive integer with $v^{\alpha^{\delta(v)}} \in \Omega_{\omega(v)}$. Define $\Omega_\lambda^{(0)} = \Omega_\lambda$ and $\Omega_\lambda^{(k+1)} = \{v \in V^{\mathrm{tra}} : v^\alpha \in \Omega_\lambda^{(k)}\}$. Equivalently, for $k \geqslant 1$, we have $\Omega_\lambda^{(k)} = \{v \in V^{\mathrm{tra}} : \omega(v) = \lambda \text{ and } \delta(v) = k\}$. Given $\lambda \in \Lambda$, we let $\kappa(\lambda)$ be the largest $k \geqslant 0$

with $\Omega_\lambda^{(k)} \neq \emptyset$. We thus obtain a natural partition

$$V = \bigsqcup_{\lambda\in\Lambda} \bigsqcup_{k=0}^{\kappa(\lambda)} \Omega_\lambda^{(k)}. \tag{6.1}$$

Cycle graphs and C_Γ^+. Let C_k denote the cycle graph on $\{1,\dots,k\}$. Various graph-theoretic properties and invariants (e.g. chromatic numbers) of C_k depend on the parity of k. Here, we encounter another instance of this phenomenon: the following elementary lemma explains the curious role of *odd*-periodic animations in Theorem 6.1.

Lemma 6.4. *Let $k \geqslant 3$. Then*

$$\det(\mathsf{C}_{\mathrm{C}_k}^+) = \begin{cases} \pm 2X_1\cdots X_k, & \text{if } k \text{ is odd},\\ 0, & \text{if } k \text{ is even.}\end{cases}$$

Proof. We order the vertices $1,\dots,k$ and the edges as $\{1,2\},\{2,3\},\dots,\{k-1,k\},\{1,k\}$. With respect to these orders, we then have

$$\mathsf{C}_\Gamma^+ = \begin{bmatrix} X_2 & X_1 & & & \\ & X_3 & X_2 & & \\ & & \ddots & \ddots & \\ & & & X_k & X_{k-1} \\ X_k & & & & X_1 \end{bmatrix}.$$

Laplace expansion along the first column yields $\det(\mathsf{C}_\Gamma^+) = (1+(-1)^{k+1})X_1\cdots X_k$. ♦

Order. For each $\lambda\in\Lambda$, we may write $\Omega_\lambda^{(0)} = \{w(\lambda,1),\dots,w(\lambda,N_\lambda)\}$ where $w(\lambda,i)^\alpha = w(\lambda,i+1)$ for $i < N_\lambda$ and $w(\lambda,N_\lambda)^\alpha = w(\lambda,1)$. We let $\preccurlyeq$ be the total order on $\Omega_\lambda^{(0)}$ with $w(\lambda,1)\prec\cdots\prec w(\lambda,N_\lambda)$. For $k=1,\dots,\kappa(\lambda)$, we choose an arbitrary total order $\preccurlyeq$ on $\Omega_\lambda^{(k)}$. Finally, we choose a total order $\preccurlyeq$ on Λ. By (6.1), each point in V corresponds uniquely to a triple (λ,k,u), where $\lambda\in\Lambda$, $k\in\{0,\dots,\kappa(\lambda)\}$, and $u\in\Omega_\lambda^{(k)}$. By ordering these triples lexicographically, we obtain a total order $\preccurlyeq$ on V.

Building a minor from an animation. Suppose that $\alpha\in\mathrm{Odd}(\Gamma)$ (resp. $\alpha\in\mathrm{Fix}(\Gamma)$). Define $V' = \mathcal{D}(\alpha)$ and $E' = \{\{v,v^\alpha\} : v\in V'\}$. We show that $|V'| = |E'|$ and that the minor $m_\Gamma^+[V' \mid E']$ (resp. $m_\Gamma^-[V' \mid E']$) is given by $\pm 2^{o\ell(\alpha)}\,\mathrm{mon}(\alpha)$.

The map $V' \xrightarrow{\Phi} E'$ which sends v to $\{v,v^\alpha\}$ is onto by construction. Suppose that $\{u,u^\alpha\} = \{v,v^\alpha\}$ for $u,v\in\mathcal{D}(\alpha)$. If $u\neq v$, then $u^\alpha = v$ and $v^\alpha = u$ whence $\{u,v\}$ is an orbit of even length, which contradicts $\alpha\in\mathrm{Odd}(\Gamma)$. Hence, Φ is bijective and $|V'| = |E'|$. By transport of structure via Φ, our existing total order on V' induces a total order on E' which we arbitrarily extend to a total order on all of E.

Note that $V \setminus V' = \Omega_0^{(1)} = \{v \in V^{\mathrm{tra}} : v^\alpha = \perp\}$. We may write $V' = \bigsqcup_{\lambda\in\Lambda} V'_\lambda$, where

$$V'_\lambda = \begin{cases} \bigsqcup_{k=2}^{\kappa(\lambda)} \Omega_\lambda^{(k)}, & \text{if } \lambda = 0,\\ \bigsqcup_{k=0}^{\kappa(\lambda)} \Omega_\lambda^{(k)}, & \text{otherwise;}\end{cases}$$

note that V'_0 might be empty but $V'_\lambda \neq \emptyset$ for $\lambda \in \Lambda^+$. (The case $\Lambda^+ = \emptyset$ is possible.) Let E'_λ denote the image of V'_λ under the bijection Φ; hence, $E' = \bigsqcup_{\lambda\in\Lambda} E'_\lambda$. For each $\lambda \in \Lambda$, the endpoints of every edge in E'_λ belong to V'_λ except when $\lambda = 0$ and the edge is of the form $\{v, v^\alpha\}$ for $v \in \Omega_0^{(2)}$; in the latter case, $v^\alpha \in \Omega_0^{(1)}$ and thus $v^\alpha \notin V'$. We thus see that $\mathsf{C}_\Gamma^\pm[V' \mid E']$ is lower block triangular with diagonal blocks given by the $\mathsf{C}_\Gamma^\pm[V'_\lambda \mid E'_\lambda]$. In particular, $m_\Gamma^\pm[V' \mid E'] = \prod_{\lambda\in\Lambda} m_\Gamma^\pm[V'_\lambda \mid E'_\lambda]$. We will further elucidate the block triangular structure in the following.

We let $D_\lambda^{(i)}$ denote the $|\Omega_\lambda^{(i)}| \times |\Omega_\lambda^{(i)}|$ diagonal matrix whose diagonal entries are given by the X_{v^α} as v ranges over $\Omega_\lambda^{(i)}$.

Nilpotent points ($\lambda = 0$). Assuming that $V'_0 \neq \emptyset$, let us consider the case $\lambda = 0$. By applying Φ, the decomposition $V'_0 = \bigsqcup_{k=2}^{\kappa(0)} \Omega_0^{(k)}$ yields a corresponding decomposition of E'_0. As we observed before, if $v \in \Omega_0^{(2)}$, then $\perp \neq v^\alpha \notin V'$. Viewing the rows of $\mathsf{C}_\Gamma^\pm[V'_0 \mid E'_0]$ as elements of $\mathbf{Z}[X_{V'_0}]E'_0$ and up to signs, the row corresponding to the edge $\{v, v^\alpha\}$ associated with $v \in \Omega_0^{(2)}$ is therefore simply $X_{v^\alpha}\mathsf{b}_v$. Next, if $v \in \Omega_0^{(k)}$ for $k \geqslant 3$, then $v^\alpha \prec v$ and the row corresponding to the edge $\{v, v^\alpha\}$ is $X_{v^\alpha}\mathsf{b}_v \pm X_v\mathsf{b}_{v^\alpha}$ (up to the sign). The submatrix $\mathsf{C}_\Gamma^\pm[V'_0 \mid E'_0]$ is therefore of the form

$$\left[\begin{array}{c|c|c} D_0^{(2)} & & \\ \hline * & D_0^{(3)} & \\ \hline & \ddots & \ddots \end{array}\right].$$

Hence, $m_\Gamma^\pm[V'_0 \mid E'_0] = \prod\limits_{v\in V'_0} X_{v^\alpha} = \mathrm{mon}(\alpha \restriction V'_0)$.

Fixed points. Next, let $\lambda \in \Lambda^+$ with $N_\lambda = 1$. In other words, $\Omega_\lambda = \{u_\lambda\}$ consists of a fixed point u_λ of α distinct from $\perp$. Arguing analogously to before, here we find that the submatrix $\mathsf{C}_\Gamma^\pm[V'_\lambda \mid E'_\lambda]$ is of the form

$$\left[\begin{array}{c|c|c|c} X_{u_\lambda} & & & \\ \hline * & D_\lambda^{(1)} & & \\ \hline & * & D_\lambda^{(2)} & \\ \hline & & \ddots & \ddots \end{array}\right].$$

The first row corresponds to the loop $\{u_\lambda\} = u_\lambda^\Phi$. Hence, we again find that $m_\Gamma^\pm[V'_\lambda \mid E'_\lambda] = \prod\limits_{v\in V'_\lambda} X_{v^\alpha} = \mathrm{mon}(\alpha \restriction V'_\lambda)$.

General periodic points, $+$-case. Let $\lambda \in \Lambda^+$ with $N_\lambda > 1$. As $\alpha \in \mathrm{Odd}(\Gamma)$, we know that $N_\lambda \geqslant 3$ is odd. In this case, we only need to consider the "positive minor" $m^+_\Gamma[V'_\lambda \mid E'_\lambda]$. (This is because we assume that $\alpha \in \mathrm{Nil}(\Gamma)$ in the $-$-case. We note that Corollary 6.9 below will imply that $m^-_\Gamma[V'_\lambda \mid E'_\lambda] = 0$ if $N_\lambda > 1$.)

Recall that $\Omega^{(0)}_\lambda = \{w(1,\lambda), \dots, w(N_\lambda, \lambda)\}$. We abbreviate $w_i = w(i,\lambda)$ and $N = N_\lambda$. Proceeding as above, we now find that $\mathsf{C}^+_\Gamma[V'_\lambda \mid E'_\lambda]$ is of the form

$$\left[\begin{array}{c|c|c|c} C & & & \\ \hline * & D^{(1)}_\lambda & & \\ \hline & * & D^{(2)}_\lambda & \\ \hline & & \ddots & \ddots \end{array}\right],$$

where C is the matrix

$$C = \begin{bmatrix} X_{w_2} & X_{w_1} & & & \\ & X_{w_3} & X_{w_2} & & \\ & & \ddots & \ddots & \\ & & & X_{w_N} & X_{w_{N-1}} \\ X_{w_N} & & & & X_{w_1} \end{bmatrix}.$$

By Lemma 6.4, we have $\det(C) = \pm 2 X_{w_1} \cdots X_{w_k}$ whence $m^\pm_\Gamma[V'_\lambda \mid E'_\lambda] = \pm 2 \prod_{v \in V'_\lambda} X_{v^\alpha} = \mathrm{mon}(\alpha \upharpoonright V'_\lambda)$.

Conclusion. In summary, we have shown the following.

Proposition 6.5.

(i) If $\alpha \in \mathrm{Fix}(\Gamma)$ (in which case $N_\lambda = 1$ for all $\lambda \in \Lambda$), then $m^-_\Gamma[V' \mid E'] = \pm\,\mathrm{mon}(\alpha)$.

(ii) If $\alpha \in \mathrm{Odd}(\Gamma)$ (in which case each N_λ is odd), then $m^+_\Gamma[V' \mid E'] = \pm 2^{o\ell(\alpha)}\,\mathrm{mon}(\alpha)$.

♦

It remains to show that, conversely, every minor $m^\pm_\Gamma[V' \mid E']$ given by $V' \subset V$ and $E' \subset E$ with $|V'| = |E'|$ is either zero, or given by $\pm 2^{o\ell(\alpha)}\,\mathrm{mon}(\alpha)$ for a suitable animation, according to the two cases in Theorem 6.1.

6.4 Minors yield animations: the connected and nondegenerate case

Let $E' \subset E$ and $V' \subset V$ with $|V'| = |E'|$. Suppose that $\mathsf{H}' = \mathsf{H}[V' \mid E']$ is connected and nondegenerate so that Lemma 5.6 is applicable. We show that the minor $m^\pm_\Gamma[V' \mid E']$ arises from a judiciously chosen animation of Γ. In showing this, we distinguish two cases reflecting the two conditions (U1) and (U2) in Lemma 5.6.

6.4.1 Case (U2): a decorated tree

Animations from rooted trees. Let T a tree on the vertex set V with chosen root $r \in V$. Let $\mathrm{d}_{\mathsf{T}}(u, v)$ denote the distance between the vertices u and v in T, i.e. the length of the unique path from u to v in T. By a **distance order** of T with respect to r, we mean a total order $\preccurlyeq$ on V such that whenever $\mathrm{d}_{\mathsf{T}}(r, u) < \mathrm{d}_{\mathsf{T}}(r, v)$ for $u, v \in V$, then we have $u \prec v$. Let $V \xrightarrow{\mathsf{pred}(\mathsf{T},r)} V$ be the partial function defined on $V \setminus \{r\}$ which sends each $v \in V \setminus \{r\}$ to its predecessor on the unique path from r to v. Clearly, $\mathsf{pred}(\mathsf{T}, r)$ is a nilpotent animation of T. By construction, we have $\mathrm{d}_{\mathsf{T}}(v^{\mathsf{pred}(\mathsf{T},r)}, r) < \mathrm{d}_{\mathsf{T}}(v, r)$ for all $v \in V' \setminus \{r\}$. By minor abuse of notation, we regard animations of subgraphs of Γ, such as $\mathsf{pred}(\mathsf{T}, r)$ when T is a spanning tree of Γ, as animations of Γ.

Suppose that we are in case (U2) of Lemma 5.6. Let $e_\circ \in E'$ be the unique loop of H'. By Lemma 5.6, $\mathsf{T} = (V', E \setminus \{e_\circ\})$ is a subtree of Γ. While $e_\circ$ is a loop of H', it may or may not be a loop of H (equivalently: Γ). We consider these two cases in turn.

Subcase: $e_\circ$ is a nonloop of H. In this case, $e_\circ$ has two distinct endpoints in H but only one of these belongs to V'. That is, $e_\circ = \{u, r\}$ where $u \in V'$ and $r \in V \setminus V'$. As T is a tree so is then the subgraph $\mathsf{T}' = (V' \cup \{r\}, E')$ of Γ. The nilpotent animation $\mathsf{pred}(\mathsf{T}', r)$ of T' and Γ turns out to give rise to the minor $m_\Gamma^{\pm}[V' \mid E']$:

Lemma 6.6. $m_\Gamma^{\pm}[V' \mid E'] = \pm\,\mathrm{mon}(\mathsf{pred}(\mathsf{T}', r))$.

Proof. Let $\preccurlyeq$ be a distance order of T' with respect to r. We arbitrarily extend $\preccurlyeq$ to a total order on V (denoted using the same symbol). Let V_i' denote the subset of V' consisting of vertices of distance i from r in T'. Given $\preccurlyeq$, we order E lexicographically as in §3.4. Write $\alpha = \mathsf{pred}(\mathsf{T}', r)$. With respect to these orders, up to changing the signs of rows, the submatrix $\mathsf{C}_\Gamma^{\pm}[V' \cup \{r\} \mid E']$ is of the form

$$\left[\begin{array}{c|c|c|c} * & D_1 & & \\ \hline & * & D_2 & \\ \hline & & \ddots & \ddots \end{array}\right],$$

where the column groups consist of $1, |V_1'|, |V_2'|, \dots$ columns, respectively, and D_i is the diagonal matrix with diagonal entries given by the X_{v^α} as v ranges over V_i' (in the given order). That is, if V_i' consists of $v_{i1} \prec v_{i2} \prec \cdots$, then $D_i = \mathrm{diag}(X_{v_{i1}^\alpha}, X_{v_{i2}^\alpha}, \dots)$. The claim follows by deleting the first column and taking the determinant to obtain $m_\Gamma^{\pm}[V' \mid E']$. ♦

Subcase: $e_\circ$ is a loop of H. In this case, $e_\circ = \{r\}$ for $r \in V'$. Using our notation from §4, we may regard $\mathsf{pred}(\mathsf{T}, r)[r \leftarrow r]$, the partial function which agrees with $\mathsf{pred}(\mathsf{T}, r)$ except that it sends r to itself, as an element of $\mathrm{Fix}(\Gamma)$.

Lemma 6.7. $m_\Gamma^{\pm}[V' \mid E'] = \pm\,\mathrm{mon}(\mathsf{pred}(\mathsf{T}, r)[r \leftarrow r])$.

Proof. Similarly to the proof of Lemma 6.6, let $\preccurlyeq$ be a distance order of T with respect to r and extend this to a total order on V. We again order edges lexicographically.

Write $\alpha = \mathsf{pred}(\mathsf{T}, r)$. Let V'_i denote the set of all vertices in V' of distance i from r. Let D_i be the diagonal matrix with entries X_{v^α} as v ranges over V'_i (in the given order). Then the submatrix $\mathsf{C}^{\pm}_{\Gamma}[V' \mid E']$ is of the form

$$\left[\begin{array}{c|c|c|c} X_r & & & \\ \hline * & D_1 & & \\ \hline & * & D_2 & \\ \hline & & \ddots & \ddots \end{array}\right],$$

where the first row corresponds to $e_\circ$. We thus obtain

$$m^{\pm}_{\Gamma}[V' \mid E'] = \pm X_r \det(D_1)\det(D_2)\cdots = \pm X_r \operatorname{mon}(\alpha) = \pm \operatorname{mon}(\alpha[r \leftarrow r]). \qquad \blacklozenge$$

6.4.2 Case (U1): a unicyclic graph

Suppose that we are in case (U1) of Lemma 5.6. Then $\Gamma = (V', E')$ is a unicyclic loopless graph. The matrices C^{+}_{Γ} and C^{-}_{Γ} behave quite differently in this case.

Lemma 6.8. *If Λ is a loopless graph with k vertices and k edges, then* $\det(\mathsf{C}^{-}_{\Lambda}) = 0$.

Proof. We may assume that $1, \dots, k$ are the vertices of Λ. As Λ is loopless, every row of C^{-}_{Λ} is of the form $X_i \mathsf{b}_j - X_j \mathsf{b}_i$. In particular, $\mathsf{C}^{-}_{\Lambda}[X_1, \dots, X_n]^\top = 0$. $\blacklozenge$

Corollary 6.9. *In case (U1) of Lemma 5.6, we have $m^{-}_{\Gamma}[V' \mid E'] = 0$.* $\blacklozenge$

Let now $\Gamma' = (V', E')$ be a unicyclic loopless subgraph of Γ. Clearly, this forces $|V'| \geqslant 3$. Let $u_1, \dots, u_r$ with $r \geqslant 3$ be distinct vertices in V' with $u_1 \sim u_2 \sim \cdots \sim u_r \sim u_1$ in Γ'. Let $U = \{u_1, \dots, u_r\}$. That is, U is the vertex set of the unique cycle within Γ'. Let $\mathsf{T} = \Gamma'/U$ be the tree obtained from Γ' by contracting all the u_i to a single vertex U (and deleting all edges on the cycle formed by U).

We order the vertices in $V' \setminus U$ by means of a distance order of T with respect to the root U. We further order the elements of U as written above, with each of them preceding each element of $V' \setminus U$. Next, we extend our order to a total order on V and we order E lexicographically.

We now construct an animation $V \xrightarrow{\alpha} V$ defined on V' as follows. As Γ' is unicyclic, for each $v \in V' \setminus U$, there exists a unique shortest path from *some* vertex in U to v. This path can be obtained from the unique path from U to v in T. For $v \in V' \setminus U$, we define v^α to be the predecessor of v on the aforementioned unique path to v. Next, we define $u_i^\alpha = u_{i+1}$ for $i < r$ and $u_r^\alpha = u_1$. Clearly, α is an animation of Γ. Intuitively, α cyclically permutes the vertices in U (having arbitrarily oriented the cycle above) and it moves vertices in $V' \setminus U$ towards U.

Let V_i' denote the set of all vertices in V' of distance i from U. Let D_i be the diagonal matrix with entries X_{v^α} as v ranges over V_i' (in the given order). We then have

$$\mathsf{C}_\Gamma^+[V' \mid E'] = \left[\begin{array}{c|c|c|c} C & & & \\ \hline * & D_1 & & \\ \hline & * & D_2 & \\ \hline & & \ddots & \ddots \end{array}\right],$$

where C is equivalent (up to relabelling of variables) to $\mathsf{C}_{\mathrm{C}_r}^+$. By construction and Lemma 6.4, we obtain the following.

Lemma 6.10.

(i) If the length r of the unique cycle in Γ' is even, then $m_\Gamma^+[V' \mid E'] = 0$.

(ii) If r is odd, then $m_\Gamma^+[V' \mid E'] = \pm 2 \operatorname{mon}(\alpha)$ and $\alpha \in \operatorname{Odd}(\Gamma)$. ♦

6.5 Minors yield animations: the general case

Let $\Gamma = (V, E)$ be a graph and let $\mathsf{H} = \mathcal{I}\!nc(\Gamma)$ be its incidence hypergraph. Let $E' \subset E$ and $V' \subset V$ with $|V'| = |E'| = k$. We are now ready to complete the proof of the remaining half of Theorem 6.1.

Proposition 6.11.

(i) If $m_\Gamma^-[V' \mid E'] \neq 0$, then there exists $\alpha \in \operatorname{Fix}(\Gamma)$ with $m_\Gamma^-[V' \mid E'] = \pm \operatorname{mon}(\alpha)$.

(ii) If $m_\Gamma^+[V' \mid E'] \neq 0$, then there exists $\alpha \in \operatorname{Odd}(\Gamma)$ with $m_\Gamma^+[V' | E'] = \pm 2^{\mathcal{o}\!\ell(\alpha)} \operatorname{mon}(\alpha)$.

Proof. Suppose that $m_\Gamma^\pm[V' \mid E'] \neq 0$. By Lemma 5.3, we may assume that $\mathsf{H}' := \mathsf{H}[V' \mid E']$ is nondegenerate. Next, we decompose $\mathsf{H}' = \mathsf{H}_1' \oplus \cdots \oplus \mathsf{H}_c'$ into connected components as in §5.2.

Writing $\mathsf{H}_j' = \mathsf{H}'[V_j' \mid E_j']$, Lemma 5.4 shows that each H_j' is square. Our results in §6.4 show that for $j = 1, \ldots, c$, we may construct an explicit animation $\alpha_j \in \operatorname{Fix}(\Gamma)$ (resp. $\alpha \in \operatorname{Odd}(\Gamma)$) with $m_\Gamma^-[V_j' \mid E_j'] = \pm \operatorname{mon}(\alpha_j)$ (resp. $m_\Gamma^+[V_j' \mid E_j'] = \pm 2^{\mathcal{o}\!\ell(\alpha_j)} \operatorname{mon}(\alpha_j)$). (We note that in the "+-case", we have $\mathcal{o}\!\ell(\alpha_j) \in \{0, 1\}$. Namely, $\mathcal{o}\!\ell(\alpha_j) = 0$ if α_j is obtained via Lemma 6.6 or Lemma 6.7 and $\mathcal{o}\!\ell(\alpha_j) = 1$ if α_j arises via Lemma 6.10.

Using our specific constructions of the α_j, the domain of definition $\mathcal{D}(\alpha_j)$ is contained in V_j'. We may thus define a partial function $V' \xrightarrow{\alpha} V'$ which agrees with α_j on V_j' for $j = 1, \ldots, c$. Clearly, $\alpha \in \operatorname{Fix}(\Gamma)$ (resp. $\alpha \in \operatorname{Odd}(\Gamma)$) if all α_j satisfy $\alpha_j \in \operatorname{Fix}(\Gamma)$ (resp. $\alpha_j \in \operatorname{Odd}(\Gamma)$). Moreover, $\mathcal{o}\!\ell(\alpha) = \mathcal{o}\!\ell(\alpha_1) + \cdots + \mathcal{o}\!\ell(\alpha_c)$. The claim thus follows since

$$m_\Gamma^\pm[V' \mid E'] = \pm \prod_{j=1}^c m_\Gamma^\pm[V_j' \mid E_j'] = \pm \prod_{j=1}^c 2^{\mathcal{o}\!\ell(\alpha_j)} \operatorname{mon}(\alpha_j) = \pm 2^{\mathcal{o}\!\ell(\alpha)} \operatorname{mon}(\alpha).$$ ♦

Proof of Theorem 6.1. Combine Proposition 6.5 and Proposition 6.11. ♦

New proof of Theorem 1.3(ii)–(iii). Combine Proposition 2.3, Proposition 3.4, and Theorem 6.1. For Theorem 1.3(ii), the base ring is $R = \mathbf{Z}[1/2]$; for Theorem 1.3(ii), it is $R = \mathbf{Z}$. ♦

7 The Reflexive Graph Modelling Theorem

In this section, we prove Theorem B by showing that, assuming that Γ is reflexive, the matrices $\mathsf{C}_\Gamma^{\pm}$ and $\mathsf{C}_{\mathscr{A}dj(\Gamma)}$ have the same rank (over $\mathbf{Q}(X_V)$) and the same ideals of minors (in $\mathbf{Z}[\frac{1}{2}][X_V]$ in the $+$-case and in $\mathbf{Z}[X_V]$ otherwise). All of this relies on our parameterisation of minors in terms of selectors and animations developed in §4 and §6.

7.1 The ranks of C_H and $\mathsf{C}_\Gamma^{\pm}$

Proposition 7.1. *Let $\mathsf{H} = (V, E, \imath)$ be a hypergraph. Then*

$$\mathrm{rk}_{\mathbf{Q}(X_V)}(\mathsf{C}_\mathsf{H}) = \#\{e \in E : \|e\|_\mathsf{H} \neq \emptyset\}.$$

Proof. By Corollary 4.2, $\mathrm{rk}_{\mathbf{Q}(X_V)}(\mathsf{C}_\mathsf{H}) = \max(\deg(\phi) : \phi \in \mathrm{Sel}(\mathsf{H}))$. Clearly, the degree of every selector of H is at most $|E| - |\mathrm{emp}(\mathsf{H})|$ and this bound is attained. ♦

Proposition 7.2. *Let $\Gamma = (V, E)$ be a graph with n vertices. Let d be the number of connected components of Γ that do* <u>*not*</u> *contain a loop. Then $\mathrm{Fix}_{n-d}(\Gamma) \neq \emptyset$ but $\mathrm{Fix}_{n-d+1}(\Gamma) = \emptyset$. In particular, $\mathrm{rk}_{\mathbf{Q}(X_V)}(\mathsf{C}_\Gamma^-) = n - d$.*

Proof. This easily reduces to the case that Γ is connected, which we now assume. Our arguments in §6.4.1 show that $\mathrm{Fix}_{n-1}(\Gamma) \neq \emptyset$. Indeed, every choice of a spanning tree T of Γ and root $r \in V$ gives rise to a (nilpotent) animation $\mathsf{pred}(\mathsf{T}, r)$ of Γ. The domain of such an animation is $V \setminus \{r\}$ whence $\emptyset \neq \mathrm{Nil}_{n-1}(\Gamma) \subset \mathrm{Fix}_{n-1}(\Gamma)$. As every nilpotent animation of Γ is necessarily undefined somewhere, we have $\mathrm{Nil}_n(\Gamma) = \emptyset$. Hence, if Γ is loopless (equivalently: $d = 1$), then $\mathrm{Fix}_n(\Gamma) = \mathrm{Nil}_n(\Gamma) = \emptyset$. On the other hand, if $r \in V$ with $r \sim r$, then $d = 0$ and $\mathsf{pred}(\mathsf{T}, r)[r \leftarrow r] \in \mathrm{Fix}_n(\Gamma)$, where T is a spanning tree of Γ as above. Of course, since the domain of any animation of Γ is a subset of V, we have $\mathrm{Fix}_{n+1}(\Gamma) = \emptyset$. The final claim follows from Corollary 6.2. ♦

Corollary 7.3. *Let Γ be a loopless graph with n vertices and c connected components. Then $\mathrm{Nil}_{n-c}(\Gamma) \neq \emptyset$ but $\mathrm{Nil}_{n-d+1}(\Gamma) = \emptyset$. In particular, $\mathrm{rk}_{\mathbf{Q}(X_V)}(\mathsf{C}_\Gamma^-) = n - c$.* ♦

Remark 7.4. The final part of Corollary 7.3 can also be deduced from [15, Lem. 3.2].

Proposition 7.5. *Let $\Gamma = (V, E)$ be a graph with n vertices. Let d be the number of connected components of Γ that do* <u>*not*</u> *contain an odd cycle; here, loops are counted amongst odd cycles. Then $\mathrm{Odd}_{n-d}(\Gamma) \neq \emptyset$ but $\mathrm{Odd}_{n-d+1}(\Gamma) = \emptyset$. In particular, $\mathrm{rk}_{\mathbf{Q}(X_V)}(\mathsf{C}_\Gamma^+) = n - d$.*

Proof. As in the proof of Proposition 7.2, we may assume that Γ is connected and we find that $\emptyset \neq \mathrm{Nil}_{n-1}(\Gamma) \subset \mathrm{Odd}_{n-1}(\Gamma)$ and $\mathrm{Nil}_n(\Gamma) = \emptyset$. If Γ does not contain any cycles of odd length, then $d = 1$ and $\mathrm{Odd}_n(\Gamma) = \mathrm{Nil}_n(\Gamma) = \emptyset$. If Γ contains a loop, then we can show that $\mathrm{Fix}_n(\Gamma) \neq \emptyset$ as in the proof of Proposition 7.2. Finally, if Γ contains a cycle of odd length which is not a loop, then we can construct $\alpha \in \mathrm{Odd}_n(\Gamma)$ using the procedure from §6.4.2. As before, $\mathrm{Odd}_{n+1}(\Gamma) = \emptyset$ and the final claim follows from Corollary 6.2. ♦

Corollary 7.6. *Let $\Gamma = (V, E)$ be a reflexive graph with n vertices. Then*

$$\mathrm{rk}_{\mathbf{Q}(X_V)}(\mathsf{C}_\Gamma^+) = \mathrm{rk}_{\mathbf{Q}(X_V)}(\mathsf{C}_\Gamma^-) = \mathrm{rk}_{\mathbf{Q}(X_V)}(\mathsf{C}_{\mathscr{A}dj(\Gamma)}) = n. \qquad ♦$$

7.2 Proof of Theorem B

Lemma 7.7. *Let* Γ *be a reflexive graph. Let* $\alpha \in \mathrm{Ani}(\Gamma)$. *Then there exists* $\beta \in \mathrm{Fix}(\Gamma)$ *with* $\mathrm{mon}(\alpha) = \mathrm{mon}(\beta)$.

Proof. Let V^{per} denote the set of α-periodic points. Let $\ell(\alpha)$ be the number of α-orbits of size > 1 on V^{per}. We proceed by induction on $\ell(\alpha)$. If $\ell(\alpha) = 0$, then $\alpha \in \mathrm{Fix}(\Gamma)$ so we may take $\beta = \alpha$. Next, let $\ell(\alpha) > 0$. Then we can find $r \geqslant 2$ and distinct points $u_1, \dots, u_r \in V$ with $u_i^{\alpha} = u_{i+1}$ for $i < r$ and $u_r^{\alpha} = u_1$. Using the notation from §4, define

$$\alpha' = \alpha[u_1 \leftarrow u_1, \dots, u_r \leftarrow u_r].$$

Since Γ is reflexive, $\alpha' \in \mathrm{Ani}(\Gamma)$. By construction, $\mathrm{mon}(\alpha') = \mathrm{mon}(\alpha)$ and $\ell(\alpha') < \ell(\alpha)$ whence the claim follows by induction. ♦

Proof of Theorem B. Let Γ have n vertices. Fix a compact DVR $\mathfrak{O}$ with *odd* residue characteristic. Let $s \in \mathbf{C}$ with $\mathrm{Re}(s) > n$ be arbitrary. By combining Proposition 2.2, Corollary 7.6, Lemma 7.7, Proposition 4.1, and Theorem 6.1, we see that each of

$$(1 - q^{-s}) W_{\Gamma}^{\pm}(q, q^{-s}) = (1 - q^{-s}) \zeta^{\mathrm{ask}}_{\mathsf{A}_{\Gamma}^{\pm}/\mathfrak{O}}(s)$$

and

$$(1 - q^{-s}) W_{\mathscr{A}dj(\Gamma)}(q, q^{-s}) = (1 - q^{-s}) \zeta^{\mathrm{ask}}_{\mathsf{A}_{\mathscr{A}dj(\Gamma)}/\mathfrak{O}}(s)$$

is given by

$$1 + (1 - q^{-1})^{-1} \int\limits_{(\mathfrak{O}V)^{\times} \times \mathfrak{P}} |z|^{s-1} \prod_{i=1}^{n} \frac{\|I_{i-1}(x)\|}{\|I_i(x) \cup z I_{i-1}(x)\|} \, \mathrm{d}\mu(x, z),$$

where $I_k = \langle \mathrm{mon}(\alpha) : \alpha \in \mathrm{Fix}_k(\Gamma) \rangle$. (Recall that $\mathrm{Ani}(\Gamma) = \mathrm{Sel}(\mathscr{A}dj(\Gamma))$.) ♦

8 Nilpotent animations

Proposition 3.4 and Corollary 6.3 suggest that by studying the nilpotent animations of a loopless graph Γ, we might learn something about the rational functions $W_{\Gamma}^{-}(X, T)$. In this section, we develop basic tools for working with and modifying nilpotent animations.

8.0 Nilpotent animations and in-forests

Nilpotent animations can be equivalently described in terms of *in-forests*. While this description is not logically required in the following, we include it for the benefit of readers who appreciate helpful pictures in graph-theoretic papers.

In-forests. The following is folklore. Let $\Gamma = (V, E)$ be a graph. An **orientation** of Γ is pair (s, t) of functions $E \to V$ such that $e = \{\mathrm{s}(e), \mathrm{t}(e)\}$ for all $e \in E$. We call $\mathrm{s}(e)$ and $\mathrm{t}(e)$ the **source** and **target** of e, respectively. The **outdegree** $\mathrm{outdeg}_\Gamma(v)$ of a vertex $v \in V$ is the number of edges $e \in E$ with $\mathrm{s}(e) = v$. A vertex v with $\mathrm{outdeg}_\Gamma(v) = 0$ is a **sink**. An **oriented graph** is a graph endowed with an orientation. A **forest** is an acyclic loopless graph. An **in-forest** is an oriented forest Φ with $\mathrm{outdeg}_\Phi(v) \leqslant 1$ for each vertex v of Φ. An **in-tree** is a connected in-forest. Every in-tree contains a unique sink.

An in-forest structure on a given forest is equivalently described by a choice of sinks, one from each connected component. In detail, let $\Phi = (V, E)$ be a forest. Let $V = V_1 \sqcup \cdots \sqcup V_c$ be the decomposition of V into the connected components of Φ. For $i = 1, \ldots, c$, choose $s_i \in V_i$. For each $v \in V_i \setminus \{s_i\}$, there exists a unique path from v to s_i in Φ. We endow Φ with an orientation as follows: for each $v \in V$, say $v \in V_i$, consider the unique path from v to s_i in Φ and orient all edges on this path towards s_i. This turns Φ into an in-forest whose sinks are precisely the s_i. Conversely, all in-forest structures on Φ arise in this fashion; cf. [18, Prop. 7.6].

Given an arbitrary graph $\Gamma = (V, E)$, by an **in-forest in** Γ, we mean an in-forest whose underlying forest is a subgraph of Γ with vertex set V. Hence, an in-forest in Γ uniquely determines and is uniquely determined by to two pieces of data:

- an acyclic set of edges $E' \subset E$ and
- a choice of sinks, one from each connected component of the forest (V, E').

Nilpotent animations of Γ and in-forests in Γ. Let $\Gamma = (V, E)$ be a graph with n vertices. As we explain in the following, nilpotent animations of Γ and in-forests in Γ are naturally in bijection. Very briefly, given $\alpha \in \mathrm{Nil}(\Gamma)$, an identity $v^\alpha = w$ (where $v, w \in V$) corresponds to an oriented edge $v \to w$ in the in-forest attached to α.

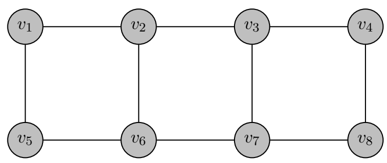
In greater detail, suppose that $\Phi = (V, E')$ is an in-forest in Γ with orientation (s, t). Define $\alpha \in \mathrm{Ani}(\Gamma)$ as follows. For $v \in V$, we have $\mathrm{outdeg}_\Phi(v) \in \{0, 1\}$. If $\mathrm{outdeg}_\Phi(v) = 0$, then we let $v^\alpha = \perp$. Otherwise, there exists a unique edge $e_v \in E'$ with $\mathrm{s}(e_v) = v$. We then let $v^\alpha = \mathrm{t}(e_v)$. As Φ is acyclic, it follows easily that α is a nilpotent animation of Γ.

Conversely, let $\alpha \in \mathrm{Nil}(\Gamma)$. Define $E' = \{\{v, v^\alpha\} : v \in \mathcal{D}(\alpha)\} \subset E$. Then $\Phi := (V, E')$ is a forest. Each connected component of Φ contains a unique vertex v with $v^\alpha = \perp$. Hence, by taking the elements of $V \setminus \mathcal{D}(\alpha)$ as our sinks, we turn Φ into an in-forest in Γ.

The preceding two constructions yield mutually inverse bijections between in-forests in Γ and nilpotent animations of Γ. If $\alpha \in \mathrm{Nil}(\Gamma)$ has degree k, then the corresponding in-forest has precisely $n - k$ connected components. The monomial associated with an in-forest is the product of the variables attached to the targets of its edges.

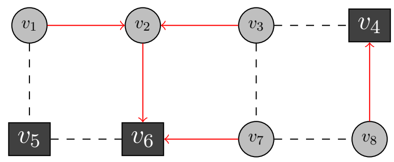
Example 8.1. Consider the following graph:

Γ:



The following is an in-forest in Γ. Here, oriented edges belong to Φ while dashed ones belong to Γ but not to Φ.

Φ:



The sinks v_4, v_5, and v_6 of Φ are drawn as squares in inverted colours. The nilpotent animation α corresponding to Φ is given by $v_1^\alpha = v_3^\alpha = v_2$, $v_2^\alpha = v_7^\alpha = v_6$, $v_8^\alpha = v_4$, and $v_4^\alpha = v_5^\alpha = v_6^\alpha = \perp$.

8.1 Using animations to order vertices

Let $\Gamma = (V, E)$ be a graph. Let $\alpha \in \mathrm{Ani}(\Gamma)$. Define a binary relation $\prec_\alpha$ on V by letting $v \prec_\alpha w$ if and only if $w = v^\alpha$ for $v, w \in V$. (Note that $v \prec_\alpha w$ forces $v^\alpha \neq \perp$.) Let $\preccurlyeq_\alpha$ be the reflexive transitive closure of $\prec_\alpha$. Then $\preccurlyeq_\alpha$ is a preorder on V. Given $v, w \in V$, we have $v \preccurlyeq_\alpha w$ if and only if there exists $n \geqslant 0$ with $v^{\alpha^n} = w$. For $v \in V$, let $\mathcal{L}_\alpha(v) = \{w \in W : w \preccurlyeq_\alpha v\}$ be the associated lower set. We record some basic properties of $\preccurlyeq_\alpha$. Recall that given a preorder $\sqsubseteq$, an element x is **maximal** if $x \sqsubseteq y$ implies $y \sqsubseteq x$.

Proposition 8.2. *Let $\Gamma = (V, E)$ be a graph, $\alpha \in \mathrm{Ani}(\Gamma)$, and $v, w \in V$. Then:*

(i) v is a $\preccurlyeq_\alpha$-maximal element of V if and only if $v^\alpha = \perp$ or v is an α-periodic point.

(ii) v and w are $\preccurlyeq_\alpha$-comparable if and only if $\mathcal{L}_\alpha(v) \cap \mathcal{L}_\alpha(w) \neq \emptyset$.

(iii) The preorder $\preccurlyeq_\alpha$ is a partial order if and only if $\alpha \in \mathrm{Fix}(\Gamma)$. If Γ is loopless, then the latter condition is equivalent to $\alpha \in \mathrm{Nil}(\Gamma)$.

(iv) If $\alpha \in \mathrm{Nil}(\Gamma)$, then $\prec_\alpha$ is the covering relation associated with $\preccurlyeq_\alpha$.

(v) Suppose that $\alpha \in \mathrm{Fix}(\Gamma)$. Then, given $v \in V$, there exists a unique $\preccurlyeq_\alpha$-maximal element $z \in V$ with $v \preccurlyeq_\alpha z$.

Proof. (i) Clear. (ii) If $v \preccurlyeq_\alpha w$, say, then $v \in \mathcal{L}_\alpha(v) \cap \mathcal{L}_\alpha(w)$. Conversely, suppose that $r \in \mathcal{L}_\alpha(v) \cap \mathcal{L}_\alpha(w)$. Then there are $m, n \geqslant 0$ with $v = r^{\alpha^m}$ and $w = r^{\alpha^n}$. Suppose, without loss of generality, that $m \leqslant n$. Then $w = v^{\alpha^{n-n}}$ and therefore $v \preccurlyeq_\alpha w$. (iii) Clear. (iv) Clear. (v) Given v, there exists a least $n \geqslant 0$ such that α sends $z := v^{\alpha^n}$ to $\perp$ or to itself. Then z is $\preccurlyeq_\alpha$-maximal with $v \preccurlyeq_\alpha z$. The uniqueness of z follows from (ii)–(iii). ♦

In the setting of Proposition 8.2(v), we write $\mathsf{last}_\alpha(v) = z$, where z is the unique $\preccurlyeq_\alpha$-maximal element of V with $v \preccurlyeq_\alpha z$. Recall from §4 that $\mathcal{D}(\alpha)$ denotes the domain of definition of α. Given a nilpotent animation α of $\Gamma = (V, E)$, the $\preccurlyeq_\alpha$-maximal elements are precisely the elements of $V \setminus \mathcal{D}(\alpha)$. We thus have the following.

Lemma 8.3. *Let $\Gamma = (V, E)$ be a graph with n vertices. Let $\alpha \in \mathrm{Nil}_k(\Gamma)$. Then the number of $\preccurlyeq_\alpha$-maximal elements of V is $n - k$.* ♦

8.2 New nilpotent animations from old ones, I

Let $\Gamma = (V, E)$ be a loopless graph. In the next section, we will carry out various types of manipulations applied to nilpotent animations with the goal of optimising various parameters. The following two lemmas are basic steps of these manipulations. The first lemma tells us precisely when redefining (or extending) a nilpotent animation gives rise to an animation which is again nilpotent. Recall that $\sim$ indicates adjacency in graphs.

Lemma 8.4 (Redefining nilpotent animations: minor surgery)**.** *Let $\alpha \in \mathrm{Nil}(\Gamma)$. Let $v, w \in V$ with $v \sim w$. Let $\beta = \alpha[v \leftarrow w] \in \mathrm{Ani}(\Gamma)$. (We emphasise that we do not require that $v^\alpha \neq \perp$ so β might be an extension of α.) Then $\beta \in \mathrm{Nil}(\Gamma)$ if and only if $w \not\preccurlyeq_\alpha v$.*

Proof. First note that since Γ is loopless and $v \sim w$, we have $v \neq w$. We will prove the lemma through a series of auxiliary claims and steps.

(a) We first claim that $\beta \notin \mathrm{Nil}(\Gamma)$ if and only if v is β-periodic.

Clearly, if v is β-periodic, then $\beta \notin \mathrm{Nil}(\Gamma)$. Conversely, suppose that $\beta \notin \mathrm{Nil}(\Gamma)$. Then there exists a β-periodic vertex, $u \in V$ say. Let $k \geqslant 1$ with $u^{\alpha^k} = u$. Since $\alpha \in \mathrm{Nil}(\Gamma)$, the vertex u is not α-periodic. We conclude that the sequence

$$u, u^\beta, u^{\beta^2}, \dots, u^{\beta^{k-1}}, u^{\beta^k} = u$$

must contain v. In particular, v is β-periodic.

(b) Next, we note that v is β-periodic if and only if $w \preccurlyeq_\beta v$. Indeed, as $v^\beta = w$, we see that v is β-periodic if and only if repeated application of β takes w to v or, equivalently, $w \preccurlyeq_\beta v$.

(c) We claim that if $w \preccurlyeq_\beta v$, then also $w \preccurlyeq_\alpha v$.

Suppose that $w \preccurlyeq_\beta v$. Let $k \geqslant 0$ be minimal with $v = w^{\beta^k}$. Since $v \neq w$, we have $k \geqslant 1$. By the minimality of k, each of $w, w^\beta, \dots, w^{\beta^{k-1}}$ is distinct from v. Thus, β acts like α on these points whence $w \preccurlyeq_\alpha v$.

(d) Suppose that $w \preccurlyeq_\alpha v$. By construction, β acts like α on the points $w, w^\alpha, \dots$ preceding v. Hence, $w \preccurlyeq_\beta v \preccurlyeq_\beta w$ whence $\beta \notin \mathrm{Nil}(\Gamma)$ by Proposition 8.2(iii). This proves the "only if" part.

(e) Suppose that $w \not\preccurlyeq_\alpha v$. By step (c), we then have $w \not\preccurlyeq_\beta v$. By step (b), v is then <u>not</u> β-periodic whence $\beta \in \mathrm{Nil}(\Gamma)$ follows from step (a). This proves the "if part". ♦

Remark 8.5. Expressed in the language of in-forests from §8.0, Lemma 8.4 asserts the following. Let Φ be an in-forest in Γ. Let v and w be adjacent vertices of Γ. Let Φ' be the oriented graph obtained from Φ by deleting, if it exists, the (necessarily unique) edge with source v and by inserting an oriented edge $v \to w$. Then Φ' is an in-forest if and only if Φ does not contain a directed path from w to v.

Given a nilpotent animation α with $v^\alpha = w$, let $z = \mathsf{last}_\alpha(w)$. Suppose that $w \neq z$ and $v \sim z \sim w$. Then the following lemma allows us to construct an explicit $\beta \in \mathrm{Nil}(\Gamma)$ with $v \prec_\beta z \prec_\beta w$ and such that $\mathrm{mon}(\alpha) = \mathrm{mon}(\beta)$.

Lemma 8.6 (Redefining nilpotent animations: bypass surgery I)**.** *Let $\alpha \in \mathrm{Nil}(\Gamma)$. Let $v, w, y, z \in V$ with $z^\alpha = \perp$. Suppose that*

$$v \prec_\alpha w \preccurlyeq_\alpha y \prec_\alpha z$$

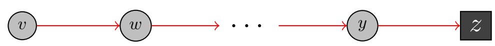
and $v \sim z \sim w$. (Note that then necessarily $\#\{v, y, z\} = 3$.) Let

$$\beta = \alpha[v \leftarrow z,\ y \leftarrow \perp,\ z \leftarrow w].$$

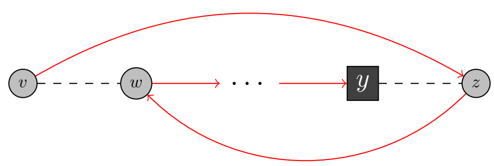
Then $\beta \in \mathrm{Nil}(\Gamma)$ and $\mathrm{mon}(\alpha) = \mathrm{mon}(\gamma)$.

Proof. This follows by repeated application of Lemma 8.4 as follows. First, let $\alpha' = \alpha[v \leftarrow \perp, y \leftarrow \perp]$. As α belongs to $\mathrm{Nil}(\Gamma)$, so does α'. Note that v, y, and z are distinct $\preccurlyeq_{\alpha'}$-maximal elements of V. By construction, we have $w \preccurlyeq_{\alpha'} y$. Proposition 8.2(v) thus shows that $w \not\preccurlyeq_{\alpha'} z$. Lemma 8.4 therefore shows that $\alpha'' = \alpha'[z \leftarrow w] \in \mathrm{Nil}(\Gamma)$. Next, v and y are distinct $\preccurlyeq_{\alpha''}$-maximal elements of V and $z \preccurlyeq_{\alpha''} y$. Again, Proposition 8.2(v) shows that $z \not\preccurlyeq_{\alpha''} v$. By applying Lemma 8.4 to α'', we thus obtain $\beta = \alpha''[v \leftarrow z] \in \mathrm{Nil}(\Gamma)$. ♦

Remark 8.7. In terms of in-forests, Lemma 8.6 asserts the following. We use the same notation as in Example 8.1. Suppose that the following is part of an in-forest Φ in Γ.



Suppose that $v \sim z \sim w$. Then the oriented graph obtained from Φ by rerouting as follows is an in-forest in Γ with the same associated monomial as Φ.



8.3 Ordering monomials relative to a point

In addition to the partial orders $\preccurlyeq_\alpha$ associated with nilpotent animations α from §8.1, we also consider partial orders $\preccurlyeq_u$ defined by a choice of a distinguished vertex u.

Let V be a finite set. Recall that $\mathfrak{O}$ denotes a compact DVR as in §1.12. Following [18, §4.2], for a subset $\mathcal{S} \subset \mathbf{R}V$, we write $\mathcal{S}(\mathfrak{O}) = \{x \in \mathfrak{O}V : \nu(x) \in \mathcal{S}\}$. We let $\cdot$ denote the inner product $x \cdot y = \sum\limits_{v \in V} x_v y_v$ on $\mathbf{R}V$. Recall that the **dual cone** of $\mathcal{S} \subset \mathbf{R}V$ is

$$\mathcal{S}^\vee = \{x \in \mathbf{R}V : x \cdot y \geqslant 0 \text{ for all } y \in \mathcal{S}\}.$$

Let $u \in V$. Define

$$\mathscr{C}_u V = \{x \in \mathbf{R}_{\geqslant 0} V : x_u \leqslant x_v \text{ for all } v \in V\}$$

We define a binary relation $\leqslant_u$ on $\mathbf{Z}V$ by letting $a \leqslant_u b$ if and only if $b - a \in (\mathscr{C}_u V)^\vee$.

Proposition 8.8.

(i) $\leqslant_u$ *is a partial order on* $\mathbf{Z}V$.

(ii) Let $a, b \in \mathbf{Z}V$ *with* $a \leqslant_u b$. *Let* $x \in \mathscr{C}_u V(\mathfrak{O})$. *Then* $x^b / x^a \in \mathfrak{O}$.

(iii) Let $a, b \in \mathbf{Z}V$ *with* $a_v \leqslant b_v$ *for all* $v \in V \setminus \{u\}$ *and* $a_u \leqslant b_u + \sum\limits_{v \in V \setminus \{u\}} (b_v - a_v)$.

Then $a \leqslant_u b$.

Proof.

(i) Only the antisymmetry of $\leqslant_u$ needs a justification. Suppose that $a \in \mathbf{Z}V$ with $a, -a \in (\mathscr{C}_u V)^\vee$. Since $\mathsf{b}_v \in \mathscr{C}_u V$ for $v \in V \setminus \{u\}$, we have $a_v = 0$ for all $v \in V \setminus \{u\}$. Next, $z := \sum_{v \in V} \mathsf{b}_v \in \mathscr{C}_u V$ and thus $a_u = a \cdot z = 0$.

(ii) For every $c \in \mathbf{Z}V$, we have $\nu(x^c) = \nu(x) \cdot c$. Hence, if $a \leqslant_u b$, then $\nu(x^{b-a}) = \nu(x) \cdot (b - a) \geqslant 0$ whence $\nu(x^a) \leqslant \nu(x^b)$.

(iii) Let $x \in \mathscr{C}_u V$ be arbitrary. Then

$$(a_u - b_u) x_u \leqslant \sum_{v \in V \setminus \{u\}} (b_v - a_v) x_u \leqslant \sum_{v \in V \setminus \{u\}} (b_v - a_v) x_v$$

whence $(b - a) \cdot x \geqslant 0$. Thus, $b - a \in (\mathscr{C}_u V)^\vee$. ♦

We also write $\leqslant_u$ for the partial order on Laurent monomials in X_V given by $X^a \leqslant_u X^b$ if and only if $a \leqslant_u b$.

Proposition 8.9. *Let* $u \in V$.

(i) Let $w \in V$. *Let* m *be an arbitrary Laurent monomial in* X_V. *Then* $X_u m \leqslant_u X_w m$. *In particular,* $X_u X_w^{-1} m \leqslant_u m$.

(ii) Let $\mathfrak{O}$ *be a compact* DVR. *Let* $x \in \mathfrak{O}V$ *with* $\nu(x_u) \leqslant \nu(x_v)$ *for all* $v \in V$ *and such that* $\prod\limits_{v \in V} x_v \neq 0$. *Let* $e, f \in \mathbf{N}_0 V$ *with* $X_V^e \leqslant_u X_V^f$. *Then* $x^e \mid x^f$.

Proof.

(i) We may assume that $u \neq w$. Write $m = X_V^e$ for $e \in \mathbf{Z}V$. Let $a = \mathsf{b}_u + e$ and $b = \mathsf{b}_w + e$. Then
   - $a_v = b_v$ for all $v \in V \setminus \{u, w\}$,
   - $a_u = e_u + 1$,

- $b_u = e_u$,
- $b_w = a_w + 1$, and
- $a_u = e_u + 1 = b_u + \sum_{v \in V \setminus \{u\}} (b_v - a_v)$.

By Proposition 8.8(iii), $a \preccurlyeq_u b$ and thus $X_u m = X_V^a \preccurlyeq_u X_V^b = X_w m$. The final claim follows by replacing m by $X_w^{-1} m$.

(ii) We have $\nu(x) \in \mathscr{C}_u V(\mathfrak{O})$ and $x^e, x^f \in \mathfrak{O} \setminus \{0\}$. The claim thus follows from Proposition 8.8(ii). ♦

8.4 New nilpotent animations from old ones, II

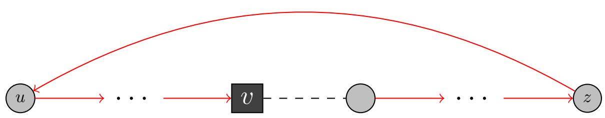
Let $\Gamma = (V, E)$ be a loopless graph. We record two lemmas in the spirit of Lemmas 8.4–8.6, but with the relation $\preccurlyeq_u$ in place of equality of monomials.

Lemma 8.10 (Redefining nilpotent animations: rerouting)**.** *Let $\alpha \in \mathrm{Nil}(\Gamma)$. Let $u, v, z \in V$ with $u \preccurlyeq_\alpha v \prec_\alpha z$ and $z^\alpha = \bot$. Suppose that $u \sim z$. Let $\beta = \alpha[v \leftarrow \bot, z \leftarrow u]$. Then $\beta \in \mathrm{Nil}(\Gamma)$, $\deg(\beta) = \deg(\alpha)$, and $\mathrm{mon}(\beta) \preccurlyeq_u \mathrm{mon}(\alpha)$.*

Proof. Let $\alpha' = \alpha[v \leftarrow \bot]$. Then $u \preccurlyeq_{\alpha'} v$ and since v and z are distinct $\preccurlyeq_{\alpha'}$-maximal elements of V, Proposition 8.2(v) implies that $u \not\preccurlyeq_{\alpha'} z$. Lemma 8.4 thus shows that $\beta = \alpha'[z \leftarrow u] = \alpha[v \leftarrow \bot, z \leftarrow u] \in \mathrm{Nil}(\Gamma)$ and $\deg(\alpha) = \deg(\beta)$. By Proposition 8.9(i), $\mathrm{mon}(\beta) = X_u X_{v^\alpha}^{-1} \,\mathrm{mon}(\alpha) \preccurlyeq_u \mathrm{mon}(\alpha)$. ♦

Remark 8.11. In terms of in-forests, Lemma 8.10 asserts the following. Suppose that the following is part of an in-forest Φ in Γ and that $u \sim z$.

Then the oriented graph Φ' obtained from Φ by rerouting as follows is an in-forest in Γ whose associated monomial is less than or equal w.r.t. u than the monomial of Φ and of the same degree.



Lemma 8.12 (Redefining nilpotent animations: bypass surgery II)**.** *Let $\alpha \in \mathrm{Nil}(\Gamma)$. Let $u, i, v, z \in V$ with $i^\alpha = v$ and $z^\alpha = \bot$. Suppose that*

$$u \preccurlyeq_\alpha i \prec_\alpha v \prec_\alpha z$$

and $u \sim v \sim z$. (Note that then necessarily $\#\{i, v, z\} = 3$.) Let

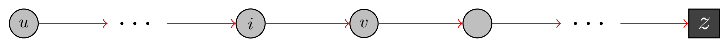
$$\beta = \alpha[i \leftarrow \bot, v \leftarrow u, z \leftarrow v].$$

Then $\beta \in \mathrm{Nil}(\Gamma)$, $\deg(\beta) = \deg(\alpha)$, and $\mathrm{mon}(\beta) \preccurlyeq_u \mathrm{mon}(\alpha)$.

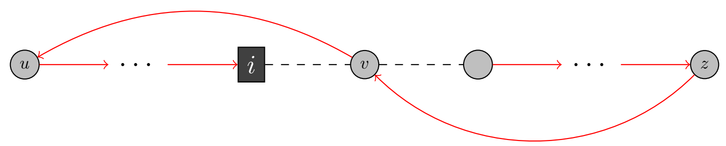
Proof. Like Lemma 8.6, this also follows by repeated application of Lemma 8.4. First, let $\alpha' = \alpha[i \leftarrow \bot, v \leftarrow \bot]$. Then i, v, and z are distinct $\preccurlyeq_{\alpha'}$-maximal elements of V. As in the proof of Lemma 8.6, we find that $u \not\preccurlyeq_{\alpha'} v$ whence $\alpha'' = \alpha'[v \leftarrow u] \in \mathrm{Nil}(\Gamma)$. Since i and z are distinct $\preccurlyeq_{\alpha''}$-maximal elements with $v \preccurlyeq_{\alpha''} u \preccurlyeq_{\alpha''} i$, we obtain $\beta = \alpha''[z \leftarrow v] \in \mathrm{Nil}(\Gamma)$. Clearly, $\deg(\alpha) = \deg(\beta)$. Finally, by Proposition 8.9(i),

$$\mathrm{mon}(\beta) = \frac{X_u X_v}{X_{i^\alpha} X_{v^\alpha}} \mathrm{mon}(\alpha) = \frac{X_u}{X_{v^\alpha}} \mathrm{mon}(\alpha) \preccurlyeq_u \mathrm{mon}(\alpha). \qquad \blacklozenge$$

Remark 8.13. In terms of in-forests, Lemma 8.6 asserts the following. Suppose that the following is part of an in-forest Φ in Γ and that $u \sim v \sim z$.



Then the oriented graph Φ' obtained from Φ by rerouting as follows is an in-forest in Γ whose associated monomial is less than or equal w.r.t. u than the monomial of Φ and of the same degree.



9 Animations of joins of graphs

Let Γ_1 and Γ_2 be loopless graphs. Let $\Gamma = \Gamma_1 \vee \Gamma_2$ be their join. Let V be the vertex set of Γ. Theorem 6.1(i) suggests that in order to prove Theorem A, we should relate the nilpotent animations of Γ to those of Γ_1 and Γ_2. In this section, we accomplish just that. Recall that by Proposition 3.4, we may express $W^-_\Gamma(q, q^{-s})$ in terms of the integral $\int_{(\mathfrak{O}V)^\times \times \mathfrak{P}} \Gamma^-(s)$. When considering the integrand in (3.2), for each $(x, z) \in (\mathfrak{O}V)^\times \times \mathfrak{P}$, there exists some vertex $u \in V$ whose associated coordinate x_u is a unit. As a major ingredient of our proof of Theorem A, instead of characterising all nilpotent animations of Γ, in this section, we exhibit a subset of *u-centred* animations (see Definition 9.1) relative to an arbitrary but fixed vertex $u \in V$, corresponding to a unit coordinate as above. For each choice of u, the key features of u-centred animations are as follows.

- Let u belong to Γ_i. Then u-centred animations of Γ arise very explicitly from nilpotent animations of Γ_i. In particular, the monomial associated with a u-centred animation can be explicitly described in terms of the monomial associated with an associated animation of Γ_i (Example 9.4 and Proposition 9.5).

- Every nilpotent animation is "dominated" by a u-centred one of the same degree (Theorem 9.3). This allows us to focus on u-centred animations only.

Let V_i be the vertex set of Γ_i. Clearly,

$$\int_{(\mathfrak{O}V)^\times \times \mathfrak{P}} \Gamma^-(s) = \int_{(\mathfrak{O}V_1)^\times \times (\mathfrak{O}V_2)^\times \times \mathfrak{P}} \Gamma^-(s) + \int_{(\mathfrak{O}V_1)^\times \times \mathfrak{P}V_2 \times \mathfrak{P}} \Gamma^-(s) + \int_{\mathfrak{P}V_1 \times (\mathfrak{O}V_2)^\times \times \mathfrak{P}} \Gamma^-(s). \quad (9.1)$$

Among the summands on the right-hand side of (9.1), the first is the easiest to analyse since it can be computed explicitly in terms of n_1 and n_2; see Lemma 11.2. Our proof of Theorem A in §11 will rely heavily on an analysis of the second and third summand in (9.1) using the machinery surrounding u-centred animations developed in the following.

9.1 Setup, centred animations, and main result

Let $\Gamma_1 = (V_1, E_1)$ and $\Gamma_2 = (V_2, E_2)$ be loopless graphs. We assume that $V_1 \cap V_2 = \emptyset$ and $V_1 \neq \emptyset \neq V_2$. Write $n_i = |V_i|$ and $V = V_1 \sqcup V_2$. Let $\Gamma = \Gamma_1 \vee \Gamma_2$ be the join of Γ_1 and Γ_2.

Definition 9.1. Let $\alpha \in \mathrm{Nil}_k(\Gamma)$. Let $u \in V$. Write $\{1, 2\} = \{i, j\}$ with $u \in V_i$. We say that α is ***u*-centred** if $|V_j \cap \mathcal{D}(\alpha)| = \min(k, n_j)$ and $v^\alpha = u$ for all $v \in V_j \cap \mathcal{D}(\alpha)$.

Remark 9.2.

(i) Whether α is u-centred generally depends on the specific representation of Γ as a join $\Gamma_1 \vee \Gamma_2$ of subgraphs Γ_1 and Γ_2. These decompositions are far from unique, as the example of a complete graph K_n with $n \geqslant 4$ shows.

(ii) We expand Definition 9.1 as follows. Let $\alpha \in \mathrm{Nil}_k(\Gamma)$. If $k \leqslant n_j$, then α is u-centred if and only if $\mathcal{D}(\alpha) \subset V_j$ and $v^\alpha = u$ for all $v \in \mathcal{D}(\alpha)$. If $k \geqslant n_j$, then α is u-centred if and only if $V_j \subset \mathcal{D}(\alpha)$ and $v^\alpha = u$ for all $v \in V_j$.

To avoid having to carry around the indices i and j all the time, in the following, we simply assume that $u \in V_1$ so that $(i, j) = (1, 2)$.

We will see in §9.2 that the minors associated with u-centred nilpotent animations arise, in an explicit fashion, from minors associated with nilpotent animations of Γ_i. The following is the main result of this section.

Theorem 9.3. *Let $\Gamma = \Gamma_1 \vee \Gamma_2$ and $u \in V$ as above. Let $\alpha \in \mathrm{Nil}_k(\Gamma)$. Then there exists $\beta \in \mathrm{Nil}_k(\Gamma)$ such that β is u-centred and $\mathrm{mon}(\beta) \preccurlyeq_u \mathrm{mon}(\alpha)$.*

9.2 Minors of centred animations

Let $\Gamma = \Gamma_1 \vee \Gamma_2$ as in §9.1. Without loss of generality, let $u \in V_1$. We show the following.

(a) Nilpotent u-centred animations of Γ of degree at most n_2 ("small") can be described explicitly. The associated minors are simply powers of X_u; see Example 9.4.

(b) Nilpotent u-centred animations of Γ of degree $k \geqslant n_2$ ("large") arise explicitly from nilpotent animations of Γ_1. The associated minors are precisely of the form $X_u^{n_2} m \cdot \mathrm{mon}(\alpha')$, where $\alpha' \in \mathrm{Nil}(\Gamma_1)$ and m is a monomial in X_{V_2}; see Proposition 9.5.

Example 9.4 (Small u-centred animations and their minors)**.** Let $0 \leqslant k \leqslant n_2$. Let $v_1, \dots, v_k \in V_2$ be distinct. Define $\alpha \in \mathrm{Nil}_k(\Gamma)$ via $\mathcal{D}(\alpha) = \{v_1, \dots, v_k\}$ and $v_i^\alpha = u$. Then α is u-centred and $\mathrm{mon}(\alpha) = X_u^k$. Conversely, every u-centred nilpotent animation of degree $k \leqslant n_2$ is of this form.

As Γ is connected and loopless with $n := n_1 + n_2$ vertices, Corollary 7.3 shows that $\mathrm{Nil}_{n-1}(\Gamma) \neq \emptyset = \mathrm{Nil}_n(\Gamma)$.

Proposition 9.5 (Minors of large u-centred animation)**.** *Let $k \geqslant n_2$.*

(i) Let $\alpha \in \mathrm{Nil}_k(\Gamma)$ be u-centred. (The existence of α forces $k \leqslant n_1 + n_2 - 1$.) Let $V_1' = \{v \in V_1 : v^\alpha \in V_1\}$. We view $\alpha' = \alpha \restriction V_1'$ as an element of $\mathrm{Nil}(\Gamma_1)$. Then

$$\mathrm{mon}(\alpha) = X_u^{n_2} \prod_{v \in V_2^{\alpha^*}} X_{v^\alpha} \cdot \mathrm{mon}(\alpha').$$

(ii) Conversely, let $k = n_2 + \ell + d \leqslant n_1 + n_2 - 1$ for $\ell, d \geqslant 0$. Let $\alpha' \in \mathrm{Nil}_\ell(\Gamma_1)$ and let m be a monomial of degree d in X_{V_2}. Then there exists a u-centred animation $\alpha \in \mathrm{Nil}_k(\Gamma)$ with

$$\mathrm{mon}(\alpha) = X_u^{n_2}\, m \cdot \mathrm{mon}(\alpha').$$

Proof.

(i) Let $v \in \mathcal{D}(\alpha)$. Then $v \in V_1$ or $v \in V_2$. If $v \in V_2$, then $v^\alpha = u$ and since $|V_2 \cap \mathcal{D}(\alpha)| = n_2$, this contributes the factor $X_u^{n_2}$. If $v \in V_1$, then either $v \in V_1'$ (if $v^\alpha \in V_1$) or $v \in V_2^{\alpha^*}$ (if $v^\alpha \in V_2$).

(ii) Let $y(1), \dots, y(d) \in V_2$ (not necessarily distinct) with $m = X_{y(1)} \cdots X_{y(d)}$. By Lemma 8.3, the number of $\leqslant_{\alpha'}$-maximal elements of V_1 is $n_1 - \ell$. Since $k = n_2 + \ell + d \leqslant n_1 + n_1 - 1$ we have $n_1 - \ell \geqslant d + 1$. Hence, there are distinct $\leqslant_{\alpha'}$-maximal elements $x(1), \dots, x(d+1) \in V_1$. We may assume that $u \leqslant_{\alpha'} x(d+1)$. Let $\alpha_0 \in \mathrm{Nil}_{n_2+\ell}(\Gamma)$ be defined by

$$v^\alpha = \begin{cases} v^{\alpha'}, & \text{if } v \in V_1, \\ u, & \text{if } u \in V_2. \end{cases}$$

Note that $x(1), \dots, x(d+1)$ are $\leqslant_{\alpha_0}$-maximal with $y(i) \leqslant_{\alpha_0} u \leqslant_{\alpha_0} x(d+1)$ for $i = 1, \dots, d$. Let $\alpha = \alpha_0[x(1) \leftarrow y(1), \dots, x(d) \leftarrow y(d)]$. Repeated application of Lemma 8.4 shows that $\alpha \in \mathrm{Nil}_k(\Gamma)$. By construction, $\mathrm{mon}(\alpha) = X_u^{n_2} m \,\mathrm{mon}(\alpha')$. ♦

9.3 Proof of Theorem 9.3

Let $\Gamma = (V, E)$ be a loopless graph. Let $u \in V$. Suppose that $V = V_1 \sqcup V_2$ such that $V_1 \neq \emptyset \neq V_2$ and $v_1 \sim v_2$ for all $v_1 \in V_1$ and $v_2 \in V_2$. Without loss of generality, suppose that $u \in V_1$. Write $n_i = |V_i|$. For $\alpha \in \mathrm{Nil}(\Gamma)$, let

$$\begin{aligned} R(\alpha) &= \#\{v \in V_2 : v^\alpha \in V_2\}, \\ L_u(\alpha) &= \#\{v \in V_2 : v^\alpha \in V_1 \setminus \{u\}\}, \text{ and} \\ M(\alpha) &= \#\{v \in V_2 : v^\alpha = u\}. \end{aligned}$$

Given $\alpha \in \mathrm{Nil}(\Gamma)$ and $v \in V$, recall that $\mathsf{last}_\alpha(v)$ denotes the unique $\preccurlyeq_\alpha$-maximal element of V with $v \preccurlyeq_\alpha \mathsf{last}_\alpha(v)$ (see §8.1). As we will see below, Theorem 9.3 will follow from an explicit procedure which, starting with an initial animation $\alpha \in \mathrm{Nil}_k(\Gamma)$, minimises $R(\alpha)$ and $L_u(\alpha)$ and which maximises $M(\alpha)$. It is based on the following three lemmas.

Lemma 9.6. *Let $\alpha \in \mathrm{Nil}_k(\Gamma)$. Then there exists $\beta \in \mathrm{Nil}_k(\Gamma)$ with $\mathrm{mon}(\beta) \preccurlyeq_u \mathrm{mon}(\alpha)$ and $R(\beta) = 0$.*

Proof. Suppose that $R(\alpha) > 0$, say $v^\alpha = w$ for $v, w \in V_2$. By induction, it suffices to find $\beta \in \mathrm{Nil}_k(\Gamma)$ with $\mathrm{mon}(\beta) \preccurlyeq_u \mathrm{mon}(\alpha)$ and $R(\beta) < R(\alpha)$.

If $u \not\preccurlyeq_\alpha v$, then Lemma 8.4 shows that we may simply take $\beta = \alpha[v \leftarrow u] \in \mathrm{Nil}_k(\Gamma)$, which indeed satisfies $\mathrm{mon}(\beta) \preccurlyeq_u \mathrm{mon}(\alpha)$ and $R(\beta) < R(\alpha)$. We may thus assume $u \preccurlyeq_\alpha v$.

Let $z = \mathsf{last}_\alpha(w)$. Suppose that $z \in V_2$ so that $u \sim z$. Let $\beta = \alpha[v \leftarrow \bot, z \leftarrow u]$. By Lemma 8.10, $\beta \in \mathrm{Nil}_k(\Gamma)$ and $\mathrm{mon}(\beta) \preccurlyeq_u \mathrm{mon}(\alpha)$. By construction, $R(\beta) < R(\alpha)$.

Suppose that $z \in V_1$ so that $w \neq z$. In this final case, the distinguished vertex u plays no role. Thus, as $w \prec_\alpha z$, there exists $y \in V$ with $v \prec_\alpha w \preccurlyeq_\alpha y \prec_\alpha z$. Since $v, w \in V_2$ and $z \in V_1$, we have $v \sim z \sim w$. We therefore obtain $\beta = \alpha[v \leftarrow z, z \leftarrow w, y \leftarrow \bot] \in \mathrm{Nil}_k(\Gamma)$ as in Lemma 8.6. In particular, $\mathrm{mon}(\alpha) = \mathrm{mon}(\beta)$ and clearly also $R(\beta) < R(\alpha)$. ♦

Lemma 9.7. *Let $\alpha \in \mathrm{Nil}_k(\Gamma)$ with $R(\alpha) = 0$. Then there exists $\beta \in \mathrm{Nil}_k(\Gamma)$ with $\mathrm{mon}(\beta) \preccurlyeq_u \mathrm{mon}(\alpha)$ and such that $L_u(\beta) = R(\beta) = 0$.*

Proof. Suppose that $R(\alpha) = 0$ and $L_u(\alpha) > 0$. It suffices to find $\beta \in \mathrm{Nil}_k(\Gamma)$ with $\mathrm{mon}(\beta) \preccurlyeq_u \mathrm{mon}(\alpha)$, $R(\beta) = 0$, and $L_u(\beta) < L_u(\alpha)$.

Let $v \in V_2$ and $w \in V_1 \setminus \{u\}$ with $v^\alpha = w$. Note that $u \sim v$ and $u \neq v$. If $u \not\preccurlyeq_\alpha v$, then by Lemma 8.4, we may simply take $\beta = \alpha[v \leftarrow u]$. Thus, suppose that $u \preccurlyeq_\alpha v$. Since $u \neq v$, there exists $i \in V$ with $u \preccurlyeq_\alpha i \prec_\alpha v$. Let $z = \mathsf{last}_\alpha(w)$. Then

$$u \preccurlyeq_\alpha i \prec_\alpha v \prec_\alpha w \preccurlyeq_\alpha z$$

Suppose that $z \in V_2$ so that $u \sim z$. Let $\beta = \alpha[v \leftarrow \bot, z \leftarrow u]$. By Lemma 8.10, $\beta \in \mathrm{Nil}_k(\Gamma)$ and $\mathrm{mon}(\beta) \preccurlyeq_u \mathrm{mon}(\alpha)$. By construction, $R(\beta) = R(\alpha) = 0$ and $L_u(\beta) < L_u(\alpha)$.

Suppose that $z \in V_1$ so that $u \sim v \sim z$. Let $\beta = \alpha[i \leftarrow \bot, v \leftarrow u, z \leftarrow v]$. By Lemma 8.12, $\beta \in \mathrm{Nil}_k(\Gamma)$ and $\mathrm{mon}(\beta) \preccurlyeq_u \mathrm{mon}(\alpha)$. Clearly, $R(\beta) = R(\alpha) = 0$ and $L_u(\beta) < L_u(\alpha)$. ♦

Lemma 9.8. *Let $\alpha \in \mathrm{Nil}_k(\Gamma)$ with $L_u(\alpha) = R(\alpha) = 0$. Then there exists $\beta \in \mathrm{Nil}_k(\Gamma)$ with $\mathrm{mon}(\beta) \preccurlyeq_u \mathrm{mon}(\alpha)$, $L_u(\beta) = R(\beta) = 0$, and $M(\beta) = \min(k, n_2)$.*

Proof. First suppose that $k \leqslant n_2$. By Example 9.4, there exists $\beta \in \mathrm{Nil}_k(\Gamma)$ with $\mathrm{mon}(\beta) = X_u^k$, $L_u(\beta) = R(\beta) = 0$ and $M(\beta) = k$. Clearly, $\mathrm{mon}(\beta) \preccurlyeq_u \mathrm{mon}(\alpha)$. Henceforth, let $k \geqslant n_2$. Suppose that $L_u(\alpha) = R(\alpha) = 0$ but $b := M(\alpha) < n_2$. It suffices to find $\beta \in \mathrm{Nil}_k(\Gamma)$ with $\mathrm{mon}(\beta) \preccurlyeq_u \mathrm{mon}(\alpha)$, $L_u(\beta) = R(\beta) = 0$, and $M(\beta) > M(\alpha)$. As $L_u(\alpha) = R(\alpha) = 0$, we have $b = M(\beta) = |\mathcal{D}(\alpha) \cap V_2|$. Let $y_1, \dots, y_b \in V_2$ be distinct with $y_i^\alpha = u$ for $i = 1, \dots, b$. Since $b < n_2$, there exists $z \in V_2 \setminus \{y_1, \dots, y_b\}$. Note that $L_u(\alpha) = R(\alpha) = 0$ and $z \neq y_i$ for $i = 1, \dots, b$ together imply that $z^\alpha = \bot$.

Suppose that $u \not\preccurlyeq_\alpha z$. Since $b < n_2 \leqslant k = \deg(\alpha)$, there exists $v \in V_1$ with $v^\alpha \neq \perp$. Let $\beta = \alpha[v \leftarrow \perp, z \leftarrow u]$. By Lemma 8.4 (applied to $\alpha[v \leftarrow \perp]$), $\beta \in \mathrm{Nil}_k(\Gamma)$. We clearly have $\mathrm{mon}(\beta) \preccurlyeq_u \mathrm{mon}(\alpha)$, $L_u(\beta) = R(\beta) = 0$ and $M(\beta) > M(\alpha)$.

Suppose that $u \preccurlyeq_\alpha z$. In this case, we may apply Lemma 8.10 with $u = v$ to obtain $\beta = \alpha[u \leftarrow \perp, z \leftarrow u] \in \mathrm{Nil}_k(\Gamma)$ with $\mathrm{mon}(\beta) \preccurlyeq_u \mathrm{mon}(\alpha)$. Clearly, $L_u(\beta) = R(\beta) = 0$ and $M(\beta) > M(\alpha)$. ♦

Proof of Theorem 9.3. Recall that $\Gamma_i = (V_i, E_i)$. Without loss of generality, we may assume that $u \in V_1$. (Hence, in the setting of §9.1, $(i, j) = (1, 2)$.) By applying Lemmas 9.6–9.8 in succession, we obtain $\beta \in \mathrm{Nil}_k(\Gamma)$ such that $\mathrm{mon}(\beta) \preccurlyeq_u \mathrm{mon}(\alpha)$, $L_u(\beta) = R(\beta) = 0$, and $M(\beta) = \min(k, n_2)$. We conclude that if $v \in V_2$, then $v^\beta \in \{u, \perp\}$ and $|\mathcal{D}(\beta) \cap V_2| = \min(k, n_2)$. Hence, β is u-centred. ♦

9.4 Bivariate monomials as minors

Let $\Gamma = \Gamma_1 \vee \Gamma_2$ as in §9.1. As before, let $V = V_1 \sqcup V_2$ and $n = n_1 + n_2$.

Lemma 9.9. *For $i = 1, 2$, let $u_i \in V_i$. Let $0 \leqslant k \leqslant n - 1$. Then there are $e_1, e_2 \geqslant 0$ with $e_1 + e_2 = k$ such that there exists $\alpha \in \mathrm{Nil}_k(\Gamma)$ with $\mathrm{mon}(\alpha) = X_{u_1}^{e_1} X_{u_2}^{e_2}$.*

Proof. Suppose, without loss of generality, that $n_1 \leqslant n_2$. If $k \leqslant n_2$, then we simply choose distinct elements $v_1, \dots, v_k \in V_2$ and let α be given by $v_i^\alpha = u_1$ for $i = 1, \dots, k$. In this case, $\mathrm{mon}(\alpha) = X_{u_1}^k$. Thus, let $n_2 \leqslant k < n = n_1 + n_2$ so that $k - n_2 < n_1$. Let $V_2 = \{v_1, \dots, v_{n_2}\}$ and let $w_1, \dots, w_{k-n_2}$ be distinct elements of $V_1 \setminus \{u_1\}$. Define $\alpha \in \mathrm{Nil}_k(\Gamma)$ via $v_i^\alpha = u_1$ for $i = 1, \dots, n_2$ and $w_j^\alpha = u_2$ for $j = 1, \dots, k - n_2$. Then $\mathrm{mon}(\alpha) = X_{u_1}^{n_2} X_{u_2}^{k-n_2}$. ♦

The following observation will be the key to computing the first summand on the right-hand side of (9.1) in Lemma 11.2.

Corollary 9.10. *Let $x \in \mathfrak{O}V$. Let $u_i \in V_i$ for $i = 1, 2$ and suppose that $x_{u_1}, x_{u_2} \in \mathfrak{O}^\times$. Then for each k with $0 \leqslant k \leqslant n_1 + n_2 - 1$, there exists $\alpha \in \mathrm{Nil}_k(\Gamma)$ with $\mathrm{mon}(\alpha)(x) \in \mathfrak{O}^\times$.* ♦

10 Adding generic rows (or columns) to matrices of linear forms

In this section, we study the effect of adding generic rows to matrices of linear forms on the minors of their ∘-duals. Our work here will play a crucial role in our proof of Theorem A in §11; see, in particular, §11.4. Apart from providing tools to be used in our proof of Theorem A, we also obtain the following result of potential independent interest.

Theorem 10.1. *Let $\mathfrak{O}$ be a compact* DVR *with residue field of size q. Let U be a finite set. Let $A \in \mathrm{M}_{n\times m}(\mathfrak{O}[X_U])$ be a matrix of linear forms. Let $\tilde{U}$ be obtained from U by adding m further symbols. Let $\tilde{A} \in \mathrm{M}_{(n+1)\times m}(\mathfrak{O}[X_{\tilde{U}}])$ be a matrix of linear forms obtained from A by adding a row populated (in some order) with the variables attached to the aforementioned symbols from $\tilde{U} \setminus U$. Then $\mathsf{Z}^{\mathrm{ask}}_{\tilde{A}/\mathfrak{O}}(T) = \mathsf{Z}^{\mathrm{ask}}_{A/\mathfrak{O}}(T) \cdot \frac{1-q^{n-m}T}{1-q^{n-m+1}T}$.*

Remark 10.2. Theorem 10.1 generalises the first part of [18, Prop. 5.24], which establishes the special case that $A = \mathsf{A}_{\mathsf{H}}$ for a hypergraph H.

For the sake of completeness, we note that the effect of adding a generic new *column* rather than row to a matrix of linear forms is easily deduced from Theorem 10.1.

Corollary 10.3. *Let the notation be as in Theorem 10.1. Let $\underset{\sim}{U}$ be obtained from U by adding n symbols. Let $\underset{\sim}{A} \in \mathrm{M}_{n\times(m+1)}(\mathfrak{O}[X_{\underset{\sim}{U}}])$ be obtained from A by adding a column containing variables attached to symbols from $\underset{\sim}{U}\setminus U$. Then $\mathsf{Z}^{\mathrm{ask}}_{\underset{\sim}{A}/\mathfrak{O}}(T) = \mathsf{Z}^{\mathrm{ask}}_{A/\mathfrak{O}}(q^{-1}T)\cdot\frac{1-q^{-1}T}{1-T}$.*

Proof. Let B be any $d \times e$ matrix of linear forms over $\mathfrak{O}$. Then [13, Lem. 2.4] yields $\mathsf{Z}^{\mathrm{ask}}_{B/\mathfrak{O}}(T) = \mathsf{Z}^{\mathrm{ask}}_{B^\top/\mathfrak{O}}(q^{d-e}T)$. Applying Theorem 10.1 to $A^\top$, we obtain $\mathsf{Z}^{\mathrm{ask}}_{\widetilde{A^\top}/\mathfrak{O}}(T) = \mathsf{Z}^{\mathrm{ask}}_{A^\top/\mathfrak{O}}(T)\cdot\frac{1-q^{m-n}T}{1-q^{m-n+1}T} = \mathsf{Z}^{\mathrm{ask}}_{A/\mathfrak{O}}(q^{m-n}T)\cdot\frac{1-q^{m-n}T}{1-q^{m-n+1}T}$. We may identify $\widetilde{A^\top} = \underset{\sim}{A}^\top$. Thus,

$$\mathsf{Z}^{\mathrm{ask}}_{\underset{\sim}{A}/\mathfrak{O}}(T) = \mathsf{Z}^{\mathrm{ask}}_{\underset{\sim}{A}^\top/\mathfrak{O}}(q^{n-m-1}T) = \mathsf{Z}^{\mathrm{ask}}_{\widetilde{A^\top}/\mathfrak{O}}(q^{n-m-1}T) = \mathsf{Z}^{\mathrm{ask}}_{A/\mathfrak{O}}(q^{-1}T)\cdot\frac{1-q^{-1}T}{1-T}. \qquad ♦$$

10.1 Some minor matrix manipulations

Let R be a ring. The group $\mathrm{GL}_n(R)\times\mathrm{GL}_m(R)$ acts on $\mathrm{M}_{n\times m}(R)$ via $A.(U,V) = U^{-1}AV$ for $A \in \mathrm{M}_{n\times m}(R)$, $U \in \mathrm{GL}_n(R)$, and $V \in \mathrm{GL}_m(R)$. Let $\simeq$ denote the corresponding equivalence relation on $\mathrm{M}_{n\times m}(R)$.

Lemma 10.4. *Let $A, B \in \mathrm{M}_{n\times m}(R)$ with $A \simeq B$. Let $r \in R$. Then*

$$\begin{bmatrix} A \\ r1_m \end{bmatrix} \simeq \begin{bmatrix} B \\ r1_m \end{bmatrix}.$$

Proof. Let $B = UAV$ for $U \in \mathrm{GL}_n(R)$, $V \in \mathrm{GL}_m(R)$. Then $\begin{bmatrix} U & 0 \\ 0 & V^{-1}\end{bmatrix}\left[\begin{smallmatrix} A \\ r1_m\end{smallmatrix}\right]V = \left[\begin{smallmatrix} b \\ r1_m\end{smallmatrix}\right]$. ♦

Recall that for a matrix A over R, we write $\mathfrak{I}_m(A)$ for the ideal of R generated by the $k\times k$ minors of A.

Lemma 10.5. *Let $A \in \mathrm{M}_{n\times m}(R)$. Let $r \in R$. Define $\widetilde{A} = \left[\begin{smallmatrix} A \\ r1_m\end{smallmatrix}\right] \in \mathrm{M}_{(n+m)\times m}(R)$. Let $0 \leqslant k \leqslant m$. Then $\mathfrak{I}_k(\widetilde{A}) = \sum\limits_{i=0}^{k} r^i\mathfrak{I}_{k-i}(A)$.*

Proof. Let $I \subset \{1,\dots,n+m\}$ and $J \subset \{1,\dots,m\}$ with $|I| = |J| = k$. Let $\widetilde{A}[I \mid J]$ be the submatrix of $\widetilde{A}$ consisting of the rows indexed by elements of I and the columns indexed by elements of J; we use analogous notation for submatrices of A. Let $I' = \{i-n : i \in I\}\cap\{1,\dots,m\}$. If $I' \not\subset J$, then $\widetilde{A}[I \mid J]$ contains a zero row whence $\det(\widetilde{A}[I \mid J]) = 0$. Thus, suppose that $I' \subset J$. Let $i = |I'| \leqslant m$ and note that

$$\det(\widetilde{A}[I \mid J]) = r^i\det(\widetilde{A}[I\setminus I' \mid J\setminus I'] = r^i\det(A[I\setminus I' \mid J\setminus I']).$$

This shows that every nonzero $k\times k$ minor of $\widetilde{A}$ is of the form r^im for $0 \leqslant i \leqslant k$ and a $(k-i)\times(k-i)$ minor m of A. Conversely, by reversing our reasoning, we find that each such element r^im arises as a minor of $\widetilde{A}$. ♦

10.2 Reminder: elementary divisors and minors

Recall that $\mathfrak{O}$ denotes a compact DVR with maximal ideal $\mathfrak{P}$, residue field size q, uniformiser $\pi \in \mathfrak{P}\backslash\mathfrak{P}^2$, normalised valuation ν, and field of fractions K. Let $A \in \mathrm{M}_{n\times m}(\mathfrak{O})$ have rank r over K. Using the structure theory of modules over the local PID $\mathfrak{O}$, we obtain a unique $\lambda(A) = (\lambda_1(A), \ldots, \lambda_r(A))$ such that $0 \leqslant \lambda_1(A) \leqslant \cdots \leqslant \lambda_r(A) < \infty$ and $A \simeq \left[\begin{smallmatrix} \mathrm{diag}(\pi^{\lambda(A)}) & 0 \\ 0 & 0 \end{smallmatrix}\right]$, where $\mathrm{diag}(\pi^{\lambda(A)}) := \mathrm{diag}(\pi^{\lambda_1(A)}, \ldots, \pi^{\lambda_r(A)})$. The $\pi^{\lambda_i(A)}$ are the elementary divisors (and invariant factors) of A.

Lemma 10.6 (Cf. [13, Lemma 4.6(ii)] or [22, §2.2])**.** *Let U be a finite set. Let $A = A(X_U) \in \mathrm{M}_{n\times m}(\mathfrak{O}[X_U])$ have rank r over $K(X_V)$. Let $x \in \mathfrak{O}U$ with $\mathrm{rk}_K(A(x)) = r$. Let $z \in \mathfrak{O} \setminus \{0\}$. Then for $i = 1, \ldots, r$, we have*

$$\frac{\|\mathfrak{I}_{i-1}(A(x))\|}{\|\mathfrak{I}_i(A(x)) \cup z\mathfrak{I}_{i-1}(A(x))\|} = q^{\min(\lambda_i(A(x)),\nu(z))}.$$

We note that the condition $\mathrm{rk}_K(A(x)) = r$ on x is satisfied outside of a null set with respect to the normalised Haar measure μ on $\mathfrak{O}U$.

10.3 Adding generic rows: elementary divisors of ∘-duals

Let R be a ring. Let U and V be finite sets with ℓ and n elements, respectively. Write $U = \{u_1, \ldots, u_\ell\}$ and $V = \{v_1, \ldots, v_n\}$. For $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant m$, and $1 \leqslant k \leqslant \ell$, let $\alpha_{ijk} \in R$. Define $A(X_U) \in \mathrm{M}_{n\times m}(R[X_U])$ and $C(X_V) \in \mathrm{M}_{\ell\times m}(R[X_V])$ via $A(X_U)_{ij} = \sum_{k=1}^{\ell} \alpha_{ijk} X_{u_k}$ and $C(X_V)_{kj} = \sum_{i=1}^{n} \alpha_{ijk} X_{v_i}$ as in (2.1). Hence, $A(X_U)$ and $C(X_V)$ are ∘-duals of each other.

Let $d \geqslant 1$. Let $\widetilde{U} = U \sqcup \{g_{rs} : 1 \leqslant r \leqslant d, 1 \leqslant j \leqslant m\}$, where the g_{rj} are distinct. Let $\widetilde{V} = V \sqcup \{w_1, \ldots, w_d\}$, where the the w_s are distinct. Write $G(d,m) = [X_{g_{rj}}] \in \mathrm{M}_{d\times m}(R[X_{\widetilde{U}}])$; hence, $G(d,m)$ is a generic $d \times n$ matrix in variables distinct from the X_u ($u \in U$). Define

$$\widetilde{A}(X_{\widetilde{U}}) = \left[\begin{array}{c} A(X_U) \\ \hline G(d,m) \end{array}\right] \in \mathrm{M}_{(n+d)\times m}(R[X_{\widetilde{U}}]) \quad \text{and}$$

$$\widetilde{C}(X_{\widetilde{V}}) = \left[\begin{array}{c} C(X_V) \\ \hline X_{w_1} 1_m \\ \vdots \\ X_{w_d} 1_m \end{array}\right] \in \mathrm{M}_{(\ell+dm)\times m}(R[X_{\widetilde{V}}]).$$

Lemma 10.7. *$\widetilde{C}(X_{\widetilde{V}})$ is a ∘-dual of $C(X_V)$.*

Proof. Ordering the elements of $\widetilde{U}$ and $\widetilde{V}$ as $u_1, \ldots, u_\ell, g_{11}, \ldots, g_{1m}, \ldots, g_{d1}, \ldots, g_{dm}$ and $v_1, \ldots, v_n, w_1, \ldots, w_d$, respectively, the claim follows by inspection. ♦

It turns out that the elementary divisors of specialisations of $\widetilde{C}(X_{\widetilde{V}})$ can be easily expressed in terms of those of specialisations of $C(X_V)$.

Lemma 10.8. *Let $\mathfrak{O}$ be a compact* DVR *with an R-algebra structure. Let r be the rank of $C(X_V)$ over K. Let $(x, y) \in \mathfrak{O}\widetilde{V} = \mathfrak{O}V \oplus \mathfrak{O}(\widetilde{V} \setminus V)$ with $\mathrm{rk}_K(C(x)) = r$ and $\prod_{i=1}^{d} y_i \neq 0$. Then*

$$\lambda_i(\widetilde{C}(x, y)) = \begin{cases} \min(\lambda_i(C(x)), \nu(y_{w_1}), \dots, \nu(y_{w_d})), & \textit{for } i = 1, \dots, r, \\ \min(\nu(y_{w_1}), \dots, \nu(y_{w_d})), & \textit{for } i = r+1, \dots, m. \end{cases}$$

Proof. Note that $\widetilde{C}(X_{\widetilde{V}})$ has rank m over $K(X_{\widetilde{V}})$. For (x, y) satisfying the conditions in the lemma, we have $\mathrm{rk}_K(\widetilde{C}(x, y)) = m$,

$$C(x) \simeq \begin{bmatrix} \mathrm{diag}(\pi^{\lambda(C(x))}) & 0 \\ 0 & 0 \end{bmatrix} \in \mathrm{M}_{\ell \times m}(\mathfrak{O}), \text{ and}$$
$$\widetilde{C}(x, y) \simeq \begin{bmatrix} \mathrm{diag}(\pi^{\lambda(\widetilde{C}(x,y))}) & 0 \\ 0 & 0 \end{bmatrix} \in \mathrm{M}_{(\ell+dm) \times m}(\mathfrak{O}).$$

On the other hand, by Lemma 10.4, we may apply elementary row operations to the $(\ell + dm) \times m$ matrix $\widetilde{C}(x, y)$ to obtain

$$\widetilde{C}(x, y) \simeq \left[\begin{array}{c} \begin{matrix} \mathrm{diag}(\pi^{\lambda(C(x))}) & 0 \\ 0 & 0 \end{matrix} \\ \hline y_{w_1} 1_m \\ \vdots \\ y_{w_d} 1_m \end{array}\right] \simeq \left[\begin{array}{c|c} \mathrm{diag}(\pi^{\min(\lambda_1(C(x)),e)}, \dots, \pi^{\min(\lambda_r(C(x)),e)}) & \\ \hline & \pi^e 1_{m-r} \\ \hline 0 & 0 \end{array}\right],$$

where $e = \min(\nu(y_{w_1}), \dots, \nu(y_{w_d}))$. The claim follows from the uniqueness of the $\lambda_i(\widetilde{C}(x, y))$; note that the exponents along the diagonal entries of the preceding matrix are nondecreasing. ♦

Corollary 10.9. *Let the assumptions be as in Lemma 10.8. Let $z \in \mathfrak{O} \setminus \{0\}$. Let $w \in \mathfrak{O}$ with $|w| = \|y_{w_1}, \dots, y_{w_d}, z\|$. Then*

$$\frac{\|\mathfrak{I}_{i-1}(\widetilde{C}(x, y))\|}{\|\mathfrak{I}_i(\widetilde{C}(x, y)) \cup z\mathfrak{I}_{i-1}(\widetilde{C}(x, y))\|} = \begin{cases} \frac{\|\mathfrak{I}_{i-1}(C(x))\|}{\|\mathfrak{I}_i(C(x)) \cup w\mathfrak{I}_{i-1}(C(x))\|}, & \textit{for } i = 1, \dots, r, \\ |w|^{-1}, & \textit{for } i = r+1, \dots, m. \end{cases}$$

Moreover, if $|w| = 1$, then $\frac{\|\mathfrak{I}_{i-1}(\widetilde{C}(x,y))\|}{\|\mathfrak{I}_i(\widetilde{C}(x,y)) \cup z\mathfrak{I}_{i-1}(\widetilde{C}(x,y))\|} = 1$.

Proof. Let $\Lambda = \min(\lambda_i(\widetilde{C}(x, y), \nu(z))$. By Lemma 10.6, the left-hand side of the displayed equation in Corollary 10.9 is q^Λ. By combining Lemmas 10.6 and 10.8, we find that q^Λ coincides with the right-hand side. ♦

10.4 Proof of Theorem 10.1

While it is possible to prove Theorem 10.1 by combining Corollary 10.9 and the integrals in Proposition 2.2, a cleaner derivation is obtained using the zeta functions attached to modules over polynomial rings from [18, §2.6].

Let V be a finite set. For $x \in \mathfrak{O}V$, we write $\mathfrak{O}_x$ for $\mathfrak{O}$ endowed with the $\mathfrak{O}[X_V]$-module structure $X_v r = x_v r$ ($v \in V$, $r \in \mathfrak{O}$). For a finitely generated $\mathfrak{O}[X_V]$-module M, define

$$\zeta_M(s) = \int\limits_{\mathfrak{O}V \times \mathfrak{O}} |z|^{s-1} \cdot |M_x \otimes_{\mathfrak{O}} \mathfrak{O}/z| \,\mathrm{d}\mu(x, z).$$

Proposition 10.10 ([18, Cor. 2.15])**.** *Let U and V be finite sets with $|U| = \ell$ and $|V| = n$. Let $A(X_U) \in \mathrm{M}_{n\times m}(\mathfrak{O}[X_U])$ be a matrix of linear forms with $\circ$-dual $C(X_V) \in \mathrm{M}_{\ell\times m}(\mathfrak{O}[X_V])$. Then $\zeta^{\mathrm{ask}}_{A(X_U)/\mathfrak{O}}(s) = (1 - q^{-1})^{-1}\zeta_{\mathrm{Coker}(C(X_V))}(s - n + m)$.*

Proof of Theorem 10.1. We work in the setting of §10.3 with $d = 1$. Dropping a superscript, we write $\widetilde{V} = V \sqcup \{w\}$ so that $\widetilde{C}(X_{\widetilde{V}}) = \begin{bmatrix} C(X_V) \\ X_w 1_m \end{bmatrix} \in \mathrm{M}_{(\ell+m)\times m}(\mathfrak{O}[X_{\widetilde{V}}])$ is a $\circ$-dual of $\widetilde{A}(X_{\widetilde{U}})$. Let $M = \mathrm{Coker}(C(X_V))$ and $\widetilde{M} = \mathrm{Coker}(\widetilde{C}(X_{\widetilde{V}}))$. By Proposition 10.10, it suffices to show that

$$\zeta_{\widetilde{M}}(s) = \frac{1 - q^{-1-s}}{1 - q^{-s}} \cdot \zeta_M(s + 1). \tag{10.1}$$

We may view $\widetilde{M}$ as the restriction of scalars of M along the ring map $\mathfrak{O}[X_{\widetilde{V}}] \to \mathfrak{O}[X_V]$ which sends X_w to 0 and which fixes each X_v ($v \in V$). We identify $\mathfrak{O}\widetilde{V} = \mathfrak{O}V \times \mathfrak{O}$, with the factor $\mathfrak{O}$ corresponding to the direct summand $\mathfrak{O}\mathsf{b}_w$ of $\mathfrak{O}\widetilde{V}$. The key observation is that for $(x, y) \in \mathfrak{O}\widetilde{V}$ and $z \in \mathfrak{O}$, we have $\widetilde{M}_{(x,y)} \otimes_{\mathfrak{O}} \mathfrak{O}/z \approx_{\mathfrak{O}} M_x \otimes_{\mathfrak{O}} \mathfrak{O}/\langle y, z\rangle$ and thus

$$\zeta_{\widetilde{M}}(s) = \int\limits_{\mathfrak{O}V \times \mathfrak{O} \times \mathfrak{O}} |z|^{s-1} |M_x \otimes_{\mathfrak{O}} \mathfrak{O}/\langle y, z\rangle| \,\mathrm{d}\mu(x, y, z).$$

We partition $\mathfrak{O}V \times \mathfrak{O} \times \mathfrak{O} = W_1 \sqcup W_2$, where

$$\begin{aligned} W_1 &= \{(x, y, z) \in \mathfrak{O}V \times \mathfrak{O} \times \mathfrak{O} : z \mid y\} \text{ and} \\ W_2 &= \{(x, y, z) \in \mathfrak{O}V \times \mathfrak{O} \times \mathfrak{O} : \pi y \mid z\}. \end{aligned}$$

To evaluate our integral over W_1, we perform a change of variables $y = zy'$ with $|\mathrm{d}y| = |z| \cdot |\mathrm{d}y'|$. We thus find that

$$\begin{aligned} \int\limits_{W_1} |z|^{s-1} |M_x \otimes_{\mathfrak{O}} \mathfrak{O}/\langle y, z\rangle| \,\mathrm{d}\mu(x, y, z) &= \int\limits_{\mathfrak{O}V \times \mathfrak{O} \times \mathfrak{O}} |z|^{s} \cdot |M_x \otimes_{\mathfrak{O}} \mathfrak{O}/z| \,\mathrm{d}\mu(x, y', z) \\ &= \zeta_M(s + 1). \end{aligned}$$

Integrating over W_2 and changing variables via $z = \pi y z'$ and $|\mathrm{d}z| = q^{-1}|y| \cdot |\mathrm{d}z'|$, using the Fubini-Tonelli theorem, we find that

$$\begin{aligned}\int_{W_2} |z|^{s-1}|M_x \otimes_{\mathfrak{O}} \mathfrak{O}/\langle y, z\rangle| \,\mathrm{d}\mu(x,y,z) &= q^{-1} \int_{\mathfrak{O}V\times\mathfrak{O}\times\mathfrak{O}} |\pi y z'|^{s-1}|M_x \otimes_{\mathfrak{O}} \mathfrak{O}/y|\,|y|\,\mathrm{d}\mu(x,y,z') \\ &= q^{-s}\int_{\mathfrak{O}} |z|^{s-1}\,\mathrm{d}\mu(z)\cdot \zeta_M(s+1).\end{aligned}$$

It is well known (and easy to prove) that $\int_{\mathfrak{O}} |z|^s \,\mathrm{d}\mu(z) = \frac{1-q^{-1}}{1-q^{-1-s}}$. Hence, we find that

$$\zeta_{\widetilde{M}}(s) = \zeta_M(s+1)\left(1 + \frac{q^{-s}(1-q^{-1})}{1-q^{-s}}\right) = \frac{1-q^{-1-s}}{1-q^{-s}}\cdot\zeta_M(s+1),$$

as claimed. ♦

11 Proof of Theorem A (and a new proof of Theorem 1.15)

As always, $\mathfrak{O}$ denotes a compact DVR with maximal ideal $\mathfrak{P}$ and residue field size q. We write $t = q^{-s}$. In the following, we assume that s is arbitrary but fixed and that $\mathrm{Re}(s)$ is sufficiently large with respect to the graphs involved.

11.1 Summary: W_Γ^- as an integral for a loopless graph Γ

Let $\Gamma = (V, E)$ be a loopless graph with n vertices and c connected components. Prior to outlining the strategy of our proof of Theorem A, we now summarise various results from [18] and the present article which provide us with a formula for W_Γ^- in terms of $\mathfrak{P}$-adic integrals. First, by Theorem 1.3(iii) and Proposition 3.4,

$$(1-t)W_\Gamma^-(q,t) = 1 + (1-q^{-1})^{-1}\int_{(\mathfrak{O}V)^\times\times\mathfrak{P}} \Gamma^-(s). \tag{11.1}$$

Next, by (3.2) and Corollary 7.3, for $W \subset \mathfrak{O}V \times \mathfrak{O}$, we have

$$\int_W \Gamma^-(s) = \int_W |z|^{s-c-1}\prod_{k=1}^{n-c}\frac{\|\mathfrak{I}_{k-1}^-\Gamma(x)\|}{\|\mathfrak{I}_k^-\Gamma(x)\cup z\mathfrak{I}_{k-1}^-\Gamma(x)\|}\,\mathrm{d}\mu(x,z). \tag{11.2}$$

Corollary 6.3 shows that for $x \in \mathfrak{O}V$, we have

$$\|\mathfrak{I}_k^-\Gamma(x)\| = \|\mathrm{mon}(\alpha)(x) : \alpha \in \mathrm{Nil}_k(\Gamma)\|. \tag{11.3}$$

Finally, Corollary 7.3 shows that $\mathfrak{I}_k^-\Gamma$ is nonempty if and only if $0 \leqslant k \leqslant n-c$.

11.2 Setup and strategy

For the remainder of this section, let $\Gamma_1 = (V_1, E_1)$ and $\Gamma_2 = (V_2, E_2)$ be loopless graphs and let $\Gamma = \Gamma_1 \vee \Gamma_2$ as in §9.1. In particular, Γ_i has n_i vertices and $n = n_1 + n_2$ is the number of vertices of Γ. Note that Γ is necessarily connected. We identify $\mathfrak{O}V = \mathfrak{O}V_1 \times \mathfrak{O}V_2$ and $X_V = (X_{V_1}, X_{V_2})$. Our proof of Theorem A is based on a series of auxiliary lemmas and claims. The first of these provides an equivalent form of Theorem A in terms of $\mathfrak{P}$-adic integrals.

Lemma 11.1. *Equation* (1.1) *holds for* $\Gamma = \Gamma_1 \vee \Gamma_2$ *if and only if*

$$\begin{aligned}\int\limits_{(\mathfrak{O}V)^\times \times \mathfrak{P}} \Gamma^-(s) = {} & \frac{(1-q^{-1})(1-q^{-n_1})(1-q^{-n_2})qt}{1-qt} \\ & + \frac{1-q^{1-n_2}t}{1-qt} \int\limits_{(\mathfrak{O}V_1)^\times \times \mathfrak{P}} \Gamma_1^-(s+n_2) \\ & + \frac{1-q^{1-n_1}t}{1-qt} \int\limits_{(\mathfrak{O}V_2)^\times \times \mathfrak{P}} \Gamma_2^-(s+n_1) \end{aligned} \tag{11.4}$$

holds for $\mathrm{Re}(s) \gg 0$.

Proof. Using (11.1), we may express $W_\Gamma^-(q,t)$ in terms of $\int\limits_{(\mathfrak{O}V)^\times \times \mathfrak{P}} \Gamma^-(s)$ and analogously for $W_{\Gamma_i}^-(q,t)$. The equivalence of (11.4) and (1.1) then follows by inspection. ♦

Our proof of Theorem A uses §§9–10 to show that the three summands on the right-hande side of (9.1) exactly match those on the right-hand side of (11.4).

For $(x, y, z) \in \mathfrak{O}V \times \mathfrak{O} = \mathfrak{O}V_1 \times \mathfrak{O}V_2 \times \mathfrak{O}$, whenever the following fraction is defined, write

$$F_k(x,y,z) := \frac{\|\mathfrak{I}_{k-1}^-\Gamma(x,y)\|}{\|\mathfrak{I}_k^-\Gamma(x,y) \cup z\mathfrak{I}_{k-1}^-\Gamma(x,y)\|}. \tag{11.5}$$

For $k = 1, \ldots, n-1$, by Lemma 10.6, we have $1 \leqslant F_k(x,y,z) \leqslant |z|^{-1}$ for almost all $(x,y) \in \mathfrak{O}V$ and all nonzero $z \in \mathfrak{O}$. Using (11.3), in the following, will express $F_k(x,y,z)$ in terms of animations of Γ.

11.3 The first summand

We now show that the first summands on the right-hand sides of (9.1) and (11.4) coincide.

Lemma 11.2. $$\int\limits_{(\mathfrak{O}V_1)^\times \times (\mathfrak{O}V_2)^\times \times \mathfrak{P}} \Gamma^-(s) = (1-q^{-1})(1-q^{-n_1})(1-q^{-n_2})\frac{qt}{1-qt}.$$

Proof. Let $0 \leqslant k \leqslant n-1$. Corollary 9.10 shows that for almost all $(x,y,z) \in (\mathfrak{O}V_1)^\times \times (\mathfrak{O}V_2)^\times \times \mathfrak{P}$, the ideal $\mathfrak{I}_k^-\Gamma$ contains a monomial m such that $m(x,y) \in \mathfrak{O}^\times$. Hence,

$F_k(x,y,z) = 1$ for $k = 1,\dots,n-1$. The claim follows from $\int_{\mathfrak{P}} |z|^s \,\mathrm{d}\mu(z) = (1 - q^{-1})q^{-1}t/(1-q^{-1}t)$ and $\int_{(\mathfrak{O}V_1)^\times \times (\mathfrak{O}V_2)^\times \times \mathfrak{P}} \Gamma^-(s) = \int_{(\mathfrak{O}V_1)^\times \times (\mathfrak{O}V_2)^\times \times \mathfrak{P}} |z|^{s-2} \,\mathrm{d}\mu(x,y,z) = \mu((\mathfrak{O}V_1)^\times)\mu((\mathfrak{O}V_2)^\times) \int_{\mathfrak{P}} |z|^{s-2} \,\mathrm{d}\mu(z)$. ♦

11.4 Towards the second summand: preparation

We derive a series of auxiliary facts which will then help us show that $\int_{(\mathfrak{O}V_1)^\times \times \mathfrak{P}V_2 \times \mathfrak{P}} \Gamma^-(s)$ coincides with the second summand on the right-hand side of (11.4).

We call $(x,y) \in \mathfrak{O}V_1 \times \mathfrak{O}V_2$ **strongly nonzero** if $x_v \neq 0 \neq y_w$ for all $v \in V_1$ and $w \in V_2$. Almost all $(x,y) \in \mathfrak{O}V_1 \times \mathfrak{O}V_2$ are strongly nonzero.

Claim 11.3. *Let $1 \leqslant k \leqslant n_2$ and let $(x,y) \in (\mathfrak{O}V_1)^\times \times \mathfrak{O}V_2$ be strongly nonzero. Then $F_k(x,y) = 1$.*

Proof. Example 9.4 shows that as long as all components x_v of x are nonzero, the ideal $\mathfrak{I}_k^- \Gamma(x,y)$ contains a unit for $0 \leqslant k \leqslant n_2$. ♦

Claim 11.4. *Let $0 \leqslant e \leqslant n_1 - 1$. Then for all strongly nonzero $(x,y) \in (\mathfrak{O}V_1)^\times \times \mathfrak{O}V_2$, we have*

$$\mathfrak{I}_{n_2+e}^- \Gamma(x,y) = \Big\langle y^b \cdot \mathrm{mon}(\alpha)(x) : 0 \leqslant d \leqslant e,\ b \in \mathbf{N}_0 V_2 \text{ with } \sum b = d,\ \alpha \in \mathrm{Nil}_{e-d}(\Gamma_1) \Big\rangle.$$

Proof. Combine Proposition 8.9, Theorem 9.3, and Proposition 9.5. ♦

Write $V_2 = \{w(1),\dots,w(n_2)\}$. Let Γ_i have m_i edges and c_i connected components. Then $\mathsf{C}_{\Gamma_i}^- \in \mathrm{M}_{m_i \times n_i}(\mathbf{Z}[X_{V_i}])$ has rank $n_i - c_i$ by Proposition 7.2. Define

$$\widetilde{\mathsf{C}}_{\Gamma_1} = \begin{bmatrix} \mathsf{C}_{\Gamma_1} \\ Y_{w(1)} 1_{n_1} \\ \vdots \\ Y_{w(n_2)} 1_{n_1} \end{bmatrix} \in \mathrm{M}_{(m_1 + n_1 n_2) \times n_1}(\mathbf{Z}[X_V]).$$

Claim 11.5. *Let $0 \leqslant e \leqslant n_1 - 1$. Then $\mathfrak{I}_{n_2+e}^- \Gamma(x,y) = \mathfrak{I}_e(\widetilde{\mathsf{C}}_{\Gamma_1}(x,y))$ for all strongly nonzero $(x,y) \in (\mathfrak{O}V_1)^\times \times \mathfrak{O}V_2$.*

Proof. This follows from Claim 11.4 and n_2 applications of Lemma 10.5, one for each block $Y_{w(j)} 1_{n_1}$ within $\widetilde{\mathsf{C}}_{\Gamma_1}$. ♦

Claim 11.6. *For all strongly nonzero $(x,y) \in (\mathfrak{O}V_1)^\times \times \mathfrak{O}V_2$ and nonzero $z \in \mathfrak{O}$, we have*

$$\prod_{k=1}^{n_1+n_2-1} F_k(x,y,z) = \prod_{e=1}^{n_1-1} \frac{\|\mathfrak{I}_{e-1}(\widetilde{\mathsf{C}}_{\Gamma_1}(x,y))\|}{\|\mathfrak{I}_e(\widetilde{\mathsf{C}}_{\Gamma_1}(x,y)) \cup z\mathfrak{I}_{e-1}(\widetilde{\mathsf{C}}_{\Gamma_1}(x,y))\|}.$$

Proof. This follows from Claims 11.3 and 11.5 and the definition of $F_k(x,y,z)$ in (11.5). ♦

Let

$$G_e(x,z) := \frac{\|\mathfrak{I}^-_{e-1}\Gamma_1(x)\|}{\|\mathfrak{I}^-_e\Gamma_1(x) \cup z\mathfrak{I}^-_{e-1}\Gamma_1(x)\|}.$$

Analogously to the case of $F_k(x,y,z)$, for $e = 1,\dots,n_1 - c_1$, by Lemma 10.6, we have $1 \leqslant G_e(x,z) \leqslant |z|^{-1}$ for almost all $x \in \mathfrak{O}V_1$ and all nonzero $z \in \mathfrak{O}$.

Claim 11.7. *For each measurable set $W \subset (\mathfrak{O}V_1)^\times \times \mathfrak{P}V_2 \times \mathfrak{P}$, we have*

$$\int\limits_W \Gamma^-(s) = \int\limits_W |z|^{s-2}\|y;z\|^{1-c_1}\prod_{e=1}^{n_1-c_1} G_e(x,\gcd(y;z))\,\mathrm{d}\mu(x,y,z).$$

Proof. Recall that $\Gamma = \Gamma_1 \vee \Gamma_2$. In particular, Γ is connected. Using (11.2) (with $c = 1$) and Claim 11.6, we obtain

$$\int\limits_W \Gamma^-(s) = \int\limits_W |z|^{s-2}\prod_{e=1}^{n_1-c_1} \frac{\|\mathfrak{I}_{e-1}(\widetilde{\mathsf{C}}_{\Gamma_1}(x,y))\|}{\|\mathfrak{I}_e(\widetilde{\mathsf{C}}_{\Gamma_1}(x,y)) \cup z\mathfrak{I}_{e-1}(\widetilde{\mathsf{C}}_{\Gamma_1}(x,y))\|}\,\mathrm{d}\mu(x,y,z).$$

We rewrite the e-indexed factors in the preceding integrand by applying Corollary 10.9 with $C = \mathsf{C}_{\Gamma_1}$, $r = n_1 - c_1$, $d = n_2$, and $\widetilde{C} = \widetilde{\mathsf{C}}_{\Gamma_1}$. Writing $g = \gcd(y;z)$ so that $|g| = \|y;z\|$, this yields

$$\begin{aligned}&\frac{\|\mathfrak{I}_{e-1}(\widetilde{\mathsf{C}}_{\Gamma_1}(x,y))\|}{\|\mathfrak{I}_e(\widetilde{\mathsf{C}}_{\Gamma_1}(x,y)) \cup z\mathfrak{I}_{e-1}(\widetilde{\mathsf{C}}_{\Gamma_1}(x,y))\|}\\ &\quad= \begin{cases} \frac{\|\mathfrak{I}_{e-1}(\mathsf{C}_{\Gamma_1}(x))\|}{\|\mathfrak{I}_e(\mathsf{C}_{\Gamma_1}(x)) \cup g\mathfrak{I}_{e-1}(\mathsf{C}_{\Gamma_1}(x))\|}, & \text{for } e = 1,\dots,n_1-c_1,\\ |g|^{-1}, & \text{for } e = n_1-c_1+1,\dots,n_1-1\end{cases}\end{aligned}$$

for almost all (x,y) and all nonzero z. The claim then follows readily. ♦

We also require the following technical and elementary lemma.

Lemma 11.8. *Let $g\colon [0,\infty) \to [0,\infty)$ be measurable. Suppose that for some $N \geqslant 0$ and all nonzero $y \in \mathfrak{P}$, we have $g(|y|) \leqslant |y|^{-N}$. For $d \geqslant 1$, let $F_d(s) = \int\limits_{\mathfrak{P}^d} \|y\|^s g(\|y\|)\,\mathrm{d}\mu(y)$. Then for $s \in \mathbf{C}$ with $\mathrm{Re}(s) \geqslant N$, we have $F_d(s) = \frac{1-q^{-d}}{1-q^{-1}}F_1(s+d-1)$.*

Proof. We proceed by induction on d, the case $d = 1$ being clear. Suppose that the claim holds for some value of d. We partition the domain of integration $\mathfrak{P}^{d+1}$ in the definition of $F_{d+1}(s)$ as $R \sqcup S$, where $R = \{y \in \mathfrak{P}^{d+1} : \|y_1,\dots,y_d\| \leqslant |y_{d+1}|\}$ and $S = \{y \in \mathfrak{P}^{d+1} : |y_{d+1}| < \|y_1,\dots,y_d\|\}$.

On R, we may write $y_i = y_{d+1}y_i'$ for $y_i' \in \mathfrak{O}$ and $i = 1, \dots, d$. A change of variables using $|\mathrm{d}y_i| = |y_{d+1}||\mathrm{d}y_i'|$ for $i = 1, \dots, d$ yields

$$\int_R \|y\|^s g(\|y\|)\,\mathrm{d}\mu(y) = \int_R |y_{d+1}|^s g(|y_{d+1}|)\,\mathrm{d}\mu(y)$$
$$= \int_{\mathfrak{O}^d \times \mathfrak{O}} |y_{d+1}|^{s+d} g(|y_{d+1}|)\,\mathrm{d}\mu(y_1', \dots, y_d', y_d) = F_1(s+d).$$

On S, write $y_{d+1} = \pi \gcd(y_1, \dots, y_d)y_{d+1}'$ for $y_{d+1}' \in \mathfrak{O}$. (Recall that $\pi \in \mathfrak{P}\setminus\mathfrak{P}^2$ denotes a uniformiser of $\mathfrak{O}$.) A change of variables using $|\mathrm{d}y_{d+1}| = q^{-1}\|y_1, \dots, y_d\||\mathrm{d}y_{d+1}'|$ yields

$$\int_S \|y\|^s g(\|y\|)\,\mathrm{d}\mu(y) = \int_S \|y_1, \dots, y_d\|^s g(\|y_1, \dots, y_d\|)\,\mathrm{d}\mu(y)$$
$$= q^{-1} \int_{\mathfrak{O}^d \times \mathfrak{O}} \|y_1, \dots, y_d\|^{s+1} g(\|y_1, \dots, y_d\|)\,\mathrm{d}\mu(y_1, \dots, y_d, y_{d+1}')$$
$$= q^{-1}F_d(s+1) = q^{-1}\frac{1-q^{-d}}{1-q^{-1}}F_1(s+d).$$

Hence, $F_{d+1}(s) = (1 + q^{-1}\frac{1-q^{-d}}{1-q^{-1}})F_1(s+d) = \frac{1-q^{-(d+1)}}{1-q^{-1}}F_1(s+d)$ as claimed. ♦

11.5 The second and third summand and a proof of Theorem A

We now deal with the second (and, by symmetry, the third) summand in (9.1).

Lemma 11.9. $$\int_{(\mathfrak{O}V_1)^\times \times \mathfrak{P}V_2 \times \mathfrak{P}} \Gamma^-(s) = \frac{1-q^{1-n_2}t}{1-qt} \int_{(\mathfrak{O}V_1)^\times \times \mathfrak{P}} \Gamma_1^-(s+n_2).$$

Proof. Consider the partition $\mathfrak{P}V_2 \times \mathfrak{P} = R \sqcup S$, where $R = \{(y,z) \in \mathfrak{P}V_2 \times \mathfrak{P} : |z| \leqslant \|y\|\}$ and $S = \{(y,z) \in \mathfrak{P}V_2 \times \mathfrak{P} : \|y\| < |z|\}$. On R, we may write $z = \gcd(y)z'$ for $z' \in \mathfrak{O}$. By taking $W = (\mathfrak{O}V_1)^\times \times R$ in Claim 11.7 and performing a change of variables using $|\mathrm{d}z| = \|y\||\mathrm{d}z'|$, we obtain

$$\int_{(\mathfrak{O}V_1)^\times \times R} \Gamma^-(s) = \int_{(\mathfrak{O}V_1)^\times \times \mathfrak{P}V_2 \times \mathfrak{O}} |z'|^{s-2}\|y\|^{s-c_1} \prod_{k=1}^{n_1-c_1} G_k(x, \gcd(y))\,\mathrm{d}\mu(x,y,z')$$
$$= \int_{\mathfrak{O}} |z'|^{s-2}\,\mathrm{d}\mu(z') \cdot \int_{(\mathfrak{O}V_1)^\times \times \mathfrak{P}V_2} \|y\|^{s-c_1} \prod_{e=1}^{n_1-c_1} G_e(x, \gcd(y))\,\mathrm{d}\mu(x,y)$$
$$= \frac{1-q^{-1}}{1-qt} \cdot \int_{(\mathfrak{O}V_1)^\times \times \mathfrak{P}V_2} \|y\|^{s-c_1} \prod_{e=1}^{n_1-c_1} G_e(x, \gcd(y))\,\mathrm{d}\mu(x,y).$$

For fixed x, we may view $\prod_{e=1}^{n_1-c_1} G_e(x,\gcd(y))$ as a function of $\|y\|$ to which Lemma 11.8 is applicable. Using the Fubini-Tonelli theorem and (11.2) (applied to Γ_1), we thus find that

$$\begin{aligned}\int\limits_{(\mathfrak{O}V_1)^\times\times R} \Gamma^-(s) &= \frac{1-q^{-1}}{1-qt}\cdot\frac{1-q^{-n_2}}{1-q^{-1}} \int\limits_{(\mathfrak{O}V_1)^\times\times\mathfrak{P}} |y|^{s+n_2-c_1-1}\prod_{k=1}^{n_1-c_1} G_k(x,y)\,\mathrm{d}\mu(x,y)\\ &= \frac{1-q^{-n_2}}{1-qt}\int\limits_{(\mathfrak{O}V_1)^\times\times\mathfrak{P}} \Gamma_1^-(s+n_2). \qquad (11.6)\end{aligned}$$

On S, we may write $y_v=\pi z y'_v$ for $v\in V_2$ and $y'_v\in\mathfrak{O}$. By taking $W=(\mathfrak{O}V_1)^\times\times S$ in Claim 11.7 and performing a change of variables using $|\mathrm{d}y_v|=q^{-1}|z||\mathrm{d}y_v|$, we obtain

$$\begin{aligned}\int\limits_{(\mathfrak{O}V_1)^\times\times S} \Gamma^-(s) &= q^{-n_2}\int\limits_{(\mathfrak{O}V_1)^\times\times\mathfrak{O}V_2\times\mathfrak{P}} |z|^{s+n_2-c_1-1}\prod_{k=1}^{n_1-c_1} G_k(x,z)\,\mathrm{d}\mu(x,y',z)\\ &= q^{-n_2}\int\limits_{(\mathfrak{O}V_1)^\times\times\mathfrak{P}} \Gamma_1^-(s+n_2). \qquad (11.7)\end{aligned}$$

Together, (11.6) and (11.7) yield

$$\begin{aligned}\int\limits_{(\mathfrak{O}V_1)^\times\times\mathfrak{P}V_2\times\mathfrak{P}} \Gamma^-(s) &= \frac{1-q^{-n_2}}{1-qt}\int\limits_{(\mathfrak{O}V_1)^\times\times\mathfrak{P}} \Gamma_1^-(s+n_2) + q^{-n_2}\int\limits_{(\mathfrak{O}V_1)^\times\times\mathfrak{P}} \Gamma_1^-(s+n_2)\\ &= \frac{1-q^{1-n_2}t}{1-qt}\int\limits_{(\mathfrak{O}V_1)^\times\times\mathfrak{P}} \Gamma_1^-(s+n_2),\end{aligned}$$

as claimed. ♦

Corollary 11.10. $$\int\limits_{\mathfrak{P}V_1\times(\mathfrak{O}V_2)^\times\times\mathfrak{P}} \Gamma^-(s) = \frac{1-q^{1-n_1}t}{1-qt}\int\limits_{(\mathfrak{O}V_2)^\times\times\mathfrak{P}} \Gamma_2^-(s+n_1).$$

Proof. Interchange the roles of Γ_1 and Γ_2 in Lemma 11.9. ♦

Proof of Theorem A. By Lemma 11.1, the conclusion of Theorem A holds for $\Gamma=\Gamma_1\vee\Gamma_2$ if and only if (11.4) holds. We then write $\int_{(\mathfrak{O}V)^\times\times\mathfrak{P}}\Gamma^-(s)$ as a sum of three integrals as in (9.1). The three summands in (9.1) agree with those in (11.4) by Lemma 11.2, Lemma 11.9, and Corollary 11.10. ♦

For a cograph Γ, the following was previously spelled out in [18, Prop. 8.4].

Corollary 11.11. *Let Γ be a loopless graph. Then* $W^-_{\Gamma\vee\bullet}(X,T)=\frac{1-X^{-1}T}{1-XT}\cdot W^-_\Gamma(X,X^{-1}T)$.

Proof. We know from [18, Table 1] that $W^-_\bullet(X,T)=1/(1-XT)$. Now apply Theorem A. ♦

11.6 A new proof of the Cograph Modelling Theorem (Theorem 1.15)

By combining Theorem A and properties of the rational functions W_{H} from [18, §5], we obtain a new (and quite short) proof of Theorem 1.15. This new proof does not use any results from [18, §§6–7], the key ingredients of the first proof of Theorem 1.15.

We first recall terminology and results from [18, §5]. Let H_1 and H_2 be hypergraphs. Following [18, §3.1], the **complete union** $\mathsf{H}_1 \circledast \mathsf{H}_2$ is obtained from the disjoint union $\mathsf{H}_1 \oplus \mathsf{H}_2$ by adjoining all vertices of H_1 to each hyperedge of H_2 and all vertices of H_2 to each hyperedge of H_1. The following results from [18] explain the effects of disjoint and complete unions on the rational functions W_{H}.

Proposition 11.12 ([18, Prop. 5.12]). *$W_{\mathsf{H}_1 \oplus \mathsf{H}_2} = W_{\mathsf{H}_1} *_T W_{\mathsf{H}_2}$ (Hadamard product in T).*

Proposition 11.13 ([18, Prop. 5.18]). *Let H_i have n_i vertices and m_i hyperedges. For $i = 1, 2$, write $y_i = X^{n_i}$ and $z_i = X^{-m_i}$. Then:*

$$W_{\mathsf{H}_1 \circledast \mathsf{H}_2} = \frac{z_1 z_2 T - 1 + W_{\mathsf{H}_1}(X, z_2 T)(1 - z_2 T)(1 - y_1 z_1 z_2 T) + W_{\mathsf{H}_2}(X, z_1 T)(1 - z_1 T)(1 - y_2 z_1 z_2 T)}{(1 - T)(1 - y_1 y_2 z_1 z_2 T)}.$$

For a hypergraph H, as in [18, §5.4], let $\mathsf{H}^{\mathbf{1}}$ be obtained from H by adding a single new hyperedge which contains all vertices. Let $\mathsf{H}^{\mathbf{0}}$ be obtained from H by adding a new hyperedge which does not contain any vertices.

Proposition 11.14 ([18, Prop. 5.24]). *$W_{\mathsf{H}^{\mathbf{1}}} = \frac{1 - X^{-1}T}{1 - T} W_{\mathsf{H}}(X, X^{-1}T)$ and $W_{\mathsf{H}^{\mathbf{0}}} = W_{\mathsf{H}}$.*

New proof of Theorem 1.15. We show that for each cograph Γ on n vertices, there exists a hypergraph H with n vertices and $n-1$ hyperedges such that $W^-_{\Gamma} = W_{\mathsf{H}}$. We proceed by structural induction. Recall that beginning with a single isolated vertex, cographs are constructed by repeatedly taking disjoint unions and joins of smaller cographs.

For the base case, if Γ consists of a single vertex, then clearly $W^-_{\Gamma} = 1/(1 - XT) = W_{\mathsf{H}}$, where H is the hypergraph on a single vertex and without hyperedges.

Let Γ_1 and Γ_2 be (co)graphs on n_1 and n_2 vertices, respectively. Suppose that H_1 and H_2 are hypergraphs such that H_i has n_i vertices and $n_i - 1$ hyperedges and such that $W^-_{\Gamma_i} = W_{\mathsf{H}_i}$. Then Proposition 11.12 and Proposition 11.14 yield

$$W_{(\mathsf{H}_1 \oplus \mathsf{H}_2)^{\mathbf{0}}} = W_{\mathsf{H}_1 \oplus \mathsf{H}_2} = W_{\mathsf{H}_1} *_T W_{\mathsf{H}_2} = W^-_{\Gamma_1} *_T W^-_{\Gamma_2} = W^-_{\Gamma_1 \oplus \Gamma_2}.$$

Noting that $(\mathsf{H}_1 \oplus \mathsf{H}_2)^{\mathbf{0}}$ has $n_1 + n_2$ vertices and $n_1 + n_2 - 1$ hyperedges, our claim thus holds for $\Gamma_1 \oplus \Gamma_2$. To show that it also holds for $\Gamma_1 \vee \Gamma_2$, let $\mathsf{H} = (\mathsf{H}_1 \circledast \mathsf{H}_2)^{\mathbf{1}}$. This hypergraph too has $n_1 + n_2$ vertices and $n_1 + n_2 - 1$ hyperedges. Let $n = n_1 + n_2$. By Proposition 11.13 (with $z_i = X^{1 - n_i}$), we have

$$\begin{aligned} W_{\mathsf{H}_1 \circledast \mathsf{H}_2} = \Big(&X^{2-n}T - 1 \\ &+ W_{\mathsf{H}_1}(X, X^{1-n_2}T)(1 - X^{1-n_2}T)(1 - X^{2-n_2}T) \\ &+ W_{\mathsf{H}_2}(X, X^{1-n_1}T)(1 - X^{1-n_1}T)(1 - X^{2-n_1}T)\Big)/((1 - T)(1 - X^2 T)). \end{aligned}$$

Using Proposition 11.14 and Theorem A, we thus find that $W_{\mathsf{H}} = W^-_{\Gamma_1 \vee \Gamma_2}$ whence our claim holds for $\Gamma_1 \vee \Gamma_2$. ♦

12 Fundamental properties of $W_\Gamma^\sharp$

In this final section, we derive an explicit graph-theoretic formula for $W_\Gamma^\sharp$ (Proposition 12.1) and derive analytic consequences (Proposition 12.3). We also show that the rational functions $W_\Gamma^\sharp$ are well behaved under joins of graphs (Proposition C). Finally, we collect formulae for $W_\Gamma^\sharp$ for all graphs on at most four vertices (Table 2). Recall that $\hat{\Gamma}$ denotes the reflexive closure of Γ. Since $W_\Gamma^\sharp = W_{\hat{\Gamma}}^\sharp$, we may assume that Γ is loopless.

An explicit formula for $W_\Gamma^\sharp$. Let $\Gamma = (V, E)$ be a (loopless) graph. The following notation matches that from [15]. For $U \subset V$, let $\mathrm{N}_\Gamma[U] \subset V$ consist of all vertices from U as well as all vertices adjacent to some vertex from U. Let $\mathrm{d}_\Gamma(U) = |\mathrm{N}_\Gamma[U] \setminus U|$, the number of vertices in $V \setminus U$ with a neighbour in U. Recall that $\widehat{\mathrm{WO}}(V)$ denotes the poset of flags of subsets of V.

Proposition 12.1. *Let $\Gamma = (V, E)$ be a (loopless) graph. Then*

$$W_\Gamma^\sharp(X^{-1}, T) = \sum_{y \in \widehat{\mathrm{WO}}(V)} (1 - X)^{|\sup(y)|} \prod_{U \in y} \frac{X^{\mathrm{d}_\Gamma(U)} T}{1 - X^{\mathrm{d}_\Gamma(U)} T}. \tag{12.1}$$

Proof. Let $\mathsf{H} = \mathcal{A}dj(\hat{\Gamma})$ so that $W_\Gamma^\sharp = W_\mathsf{H}$. Using the notation from Theorem 1.14, as $\hat{\Gamma}$ is reflexive, for each $U \subset V$, we have $\check{U} = \mathrm{N}_\Gamma[U]$. Now apply Theorem 1.14. ♦

Remark 12.2. In [15], the cardinalities of the set $\mathrm{N}_\Gamma[U]$ featured crucially in an explicit formula [15, Cor. B] for the coefficient of T in $W_\Gamma^-(X, T)$. At present, no explicit combinatorial formula for W_Γ^- akin to (12.1) is known; see [18, Question 1.8(iii)].

Local poles. Let H be a hypergraph. Theorem 1.14 shows that W_H can be written in the form

$$W_\Gamma^\sharp = \frac{f(X, T)}{\prod_{i=1}^{N} (1 - X^{a_i} T)} \tag{12.2}$$

for $f(X, T) \in \mathbf{Z}[X^{\pm 1}, T]$ and $a_1, \ldots, a_N \in \mathbf{Z}$. For any graph Γ, the same conclusion holds for $W_\Gamma^\sharp$. By Theorem 1.15, it also holds for W_Γ^- if Γ is a cograph. In any case, we may assume that $f(X, X^{-a_i}) \neq 0$ for $i = 1, \ldots, N$. It is then easy to see that the representation in (12.2) is unique up to the order of the a_i. We refer to the integers $a_1, \ldots, a_N$ as the **local poles** of W_H; multiplicities of local poles are understood in the evident way. The local poles of W_H are precisely the real parts of the poles of the meromorphic function $W_\mathsf{H}(q, q^{-s})$, where $q > 1$ is arbitrary.

Even for cographs Γ, the local poles of W_Γ^- remain mysterious. In particular, positive and negative local poles can arise and no single number appears as a universal local pole of all W_Γ^-; see [18, Table 2]. In contrast, the $W_\Gamma^\sharp$ are much better behaved.

Proposition 12.3. *Let $\Gamma = (V, E)$ be a (loopless) graph. Then:*

(a) Each local pole of $W_\Gamma^\sharp$ is nonpositive.

(b) *Let Γ have c connected components. Then $W^\sharp_\Gamma$ has a pole of order $c+1$ at $T=1$. That is, $(1-T)^{c+1}W^\sharp_\Gamma$ is regular at $T=1$ and $(1-T)^{c+1}W^\sharp_\Gamma\Big|_{T\leftarrow 1} \neq 0$.*

Proof. The first part follows from Proposition 12.1 since $\mathrm{d}_\Gamma(U) \geqslant 0$ for each $U \subset V$. It is easy to see that $\mathrm{d}_\Gamma(U) = 0$ is equivalent to U being a disjoint union of (zero or more) connected components of Γ. In particular, if $U_0 \subset \dots \subset U_r \subset V$ with $\mathrm{d}_\Gamma(U_i) = 0$ for $i = 0,\dots,r$, then $r \leqslant c$. This proves that $(1-T)^{c+1}W^\sharp_\Gamma$ is regular at $T=1$. Let $\mathcal{F} \subset \widehat{\mathrm{WO}}(V)$ consist of those flags $y = (U_0 \subsetneq U_1 \subsetneq \dots \subsetneq U_r)$ of subsets of V such that precisely $c+1$ of the U_i satisfy $\mathrm{d}_\Gamma(U_i) = 0$. Such a flag y necessarily satisfies $U_0 = \emptyset$ and $\sup(y) = U_r = V$, in addition to $r \geqslant c$. The elements of $\mathcal{F}$ are precisely the flags that contribute nonzero summands to $(1-T)^{c+1}W^\sharp_\Gamma\Big|_{T\leftarrow 1}$. It remains to rule out possible cancellations. Write $n = |V|$. Evaluating at $X = 1/2$, we obtain

$$(1-T)^{c+1}W^\sharp_\Gamma(2,T)\Big|_{T\leftarrow 1} = \sum_{y\in\mathcal{F}} 2^{-n} \prod_{\substack{U\in y\\ \mathrm{d}_\Gamma(U)>0}} \frac{2^{-\mathrm{d}_\Gamma(U)}T}{1-2^{-\mathrm{d}_\Gamma(U)}T}. \tag{12.3}$$

Given $y \in \mathcal{F}$, let $f(y) = \#\{U \in y : \mathrm{d}_\Gamma(U) > 0\} \geqslant 0$. Viewed as a power series in T, the coefficient of $T^{f(y)}$ of the summand corresponding to y on the right-hand side of (12.3) is positive and all other coefficients are nonnegative. We conclude that the right-hand side of (12.3) is nonzero. ♦

Remark 12.4. The description of the rational number given in (12.3) is reminiscent of the definition of the constant c_d in [20, (6.1)]. The latter constant occurs as a special value of the reduced and topological subgroup zeta functions of the free class-2-nilpotent groups of rank d; cf. [20, Thms. 6.8 and 6.11]. In both contexts, we lack a conceptual interpretation of these rational numbers: a group-theoretic one in the case of c_d, a graph-theoretic one in the current case.

Joins and disjoint unions.

Proof of Proposition C. Recall that $\hat{\Gamma}$ denotes the reflexive closure of a graph Γ. Clearly, $\widehat{\Gamma_1 \oplus \Gamma_2} = \hat{\Gamma}_1 \oplus \hat{\Gamma}_2$ and $\widehat{\Gamma_1 \vee \Gamma_2} = \hat{\Gamma}_1 \vee \hat{\Gamma}_2$. Using the notation from [18, §3.1], we thus have $\mathcal{A}\!dj(\widehat{\Gamma_1 \oplus \Gamma_2}) = \mathcal{A}\!dj(\hat{\Gamma}_1) \oplus \mathcal{A}\!dj(\hat{\Gamma}_2)$ and $\mathcal{A}\!dj(\widehat{\Gamma_1 \vee \Gamma_2}) = \mathcal{A}\!dj(\hat{\Gamma}_1) \circledast \mathcal{A}\!dj(\hat{\Gamma}_2)$ (complete union). The first claim follows from Proposition 11.12 and the second from [18, Cor. 5.18] with $m_i = n_i$. ♦

Formulae for small graphs. Table 2 lists the rational functions $W^\sharp_\Gamma$ for all loopless graphs on at most four vertices. Table 2 is a sequel to [18, Table 1] which lists W^+_Γ and W^-_Γ for the same class of graphs. Formulae for $W^\sharp_\Gamma$ for all 1252 graphs on at most seven vertices are available on the first author's home page. These functions were computed using Proposition 12.1. For graphs on at most six vertices, they agree with computations performed using Zeta [16]. (For some graphs on seven vertices, using the algorithms from [18] implemented in Zeta to compute $W^\sharp_\Gamma$ requires a significant amount of memory which renders these computations impractical on a typical desktop computer.)

Some infinite families of graphs. Some of the rational functions in Table 2 were previously known in the sense that they follow from existing results in the literature. First, let K_n and Δ_n denote the complete and edgeless graph on n vertices, respectively. By [13, Prop. 1.5], we have $W^\sharp_{\mathrm{K}_d} = \frac{1-X^{-d}T}{(1-T)^2}$. Using a result due to Brenti [3, Thm 3.4], [13, Cor. 5.17] provides an explicit formula for $W^\sharp_{\Delta_n}$ in terms of permutation statistics on the hyperoctahedral group $\mathrm{B}_n = \{\pm 1\} \wr \mathrm{S}_n$. More generally, [6, Cor. 5.11] provides an explicit formula for $W^\sharp_{\mathrm{K}_{d_1} \oplus \cdots \oplus \mathrm{K}_{d_n}}$ in terms of permutations statistics on $(n+1)$-coloured permutations on n letters. This includes the aforementioned known formulae for $W^\sharp_{\mathrm{K}_d}$ ($n = 1$, $d_1 = d$) and $W^\sharp_{\Delta_n}$ ($d_1 = \cdots = d_n = 1$) as special cases. In this way, 10 of the 18 formulae in Table 2 are in fact explained by [6].

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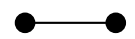
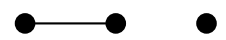
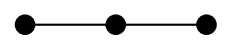
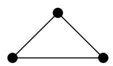
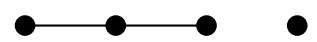
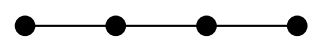
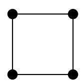
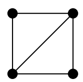
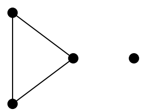
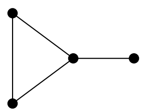
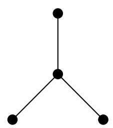
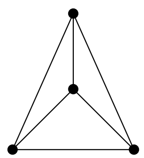
Γ	$W^{\sharp}_{\Gamma}(X,T)$
	$\frac{1-X^{-1}T}{(1-T)^2}$
	$\frac{1+X^{-2}T-4X^{-1}T+T+X^{-2}T^2}{(1-T)^3}$
	$\frac{1-X^{-2}T}{(1-T)^2}$
	$\frac{1-X^{-3}T+6X^{-2}T-12X^{-1}T+4T+T^2-4X^{-3}T^2+12X^{-2}T^2-6X^{-1}T^2-X^{-3}T^3}{(1-T)^4}$
	$\frac{1+X^{-3}T-2X^{-2}T-2X^{-1}T+T+X^{-3}T^2}{(1-T)^3}$
	$\frac{1+X^{-3}T-4X^{-2}T+X^{-1}T+X^{-4}T^2}{(1-X^{-1}T)(1-T)^2}$
	$\frac{1-X^{-3}T}{(1-T)^2}$
	$(1+X^{-4}T-8X^{-3}T+24X^{-2}T-32X^{-1}T+11T+11X^{-4}T^2-56X^{-3}T^2+96X^{-2}T^2-56X^{-1}T^2$ $+11T^2+11X^{-4}T^3-32X^{-3}T^3+24X^{-2}T^3-8X^{-1}T^3+T^3+X^{-4}T^4)/(1-T)^5$
	$\frac{1-X^{-4}T+4X^{-3}T-2X^{-2}T-8X^{-1}T+4T-4X^{-4}T^2+8X^{-3}T^2+2X^{-2}T^2-4X^{-1}T^2+T^2-X^{-4}T^3}{(1-T)^4}$
	$\frac{1+X^{-4}T-4X^{-2}T+T+X^{-4}T^2}{(1-T)^3}$
	$\frac{1-X^{-4}T+6X^{-3}T-10X^{-2}T+T-2X^{-5}T^2+X^{-4}T^2+8X^{-3}T^2+X^{-2}T^2-2X^{-1}T^2+X^{-6}T^3-10X^{-4}T^3+6X^{-3}T^3-X^{-2}T^3+X^{-6}T^4}{(1-X^{-1}T)^2(1-T)^3}$
	$\frac{1+X^{-4}T-6X^{-2}T+2X^{-1}T-2X^{-5}T^2+6X^{-4}T^2-X^{-2}T^2-X^{-6}T^3}{(1-X^{-1}T)^2(1-T)^2}$
	$\frac{1+3X^{-4}T-8X^{-3}T+3X^{-2}T+X^{-6}T^2}{(1-X^{-2}T)(1-T)^2}$
	$\frac{1+X^{-4}T-4X^{-3}T+X^{-2}T+X^{-6}T^2}{(1-X^{-2}T)(1-T)^2}$
	$\frac{1+X^{-4}T-2X^{-3}T-2X^{-1}T+T+X^{-4}T^2}{(1-T)^3}$
	$\frac{1+X^{-4}T-2X^{-3}T-2X^{-2}T+X^{-1}T+X^{-5}T^2}{(1-X^{-1}T)(1-T)^2}$
	$\frac{1-X^{-4}T+6X^{-3}T-12X^{-2}T+4X^{-1}T-4X^{-5}T^2+12X^{-4}T^2-6X^{-3}T^2+X^{-2}T^2-X^{-6}T^3}{(1-X^{-1}T)^2(1-T)^2}$
	$\frac{1-X^{-4}T}{(1-T)^2}$

Table 2: $W^{\sharp}_{\Gamma}$ for all graphs on at most four vertices