ANALYTIC PROPERTIES OF REPRESENTATION ZETA FUNCTION OF GROUPS OF TYPE A_2

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Abstract. We study analytic properties of the representation zeta functions of arithmetic groups of type A_2, such as $\mathsf{SL}_3(\mathbb{Z})$. In particular, we uncover further poles of these functions and determine a natural boundary for their meromorphic continuation beyond their abscissa of convergence. We analyse both the number field and function field case.

1. Introduction

1.1. **Euler products: meromorphic continuations and natural boundaries.** L-functions in the Selberg class have an Euler product and an analytic continuation to the entire complex plane with a pole at most at $s = 1$. Many natural L-functions come as an Euler product because of some underlying local-global principle, but the analytic continuation is by no means clear, either because it is hard to prove (which is one of the major obstacles in the Langlands program), or because it is simply not true.

There is no reason that a random Euler product should have analytic continuation, and even if it has some structure, this defines rarely an entire function. The prototypical result goes back to Estermann [9] who proved that an Euler product of the shape $\prod_p h(p^{-s})$ with $h \in \mathbb{Z}[x]$ satisfying $h(0) = 1$ has a natural boundary at $\Re s = 0$ unless h is a product of cyclotomic polynomials. This has been generalized in various ways, most notably in [7, Ch. 5] to Euler products of the form

$$\prod_p h(p, p^{-s}) \tag{1.1}$$

for a bivariate polynomial $h \in \mathbb{Z}[x, y]$. While the theory developed in [7] is not exhaustive, it indicates that such Euler products typically have—provably or conjecturally—a natural boundary a little bit to the left of the abscissa of absolute convergence, and there should be a recipe to read it off from the shape of h. Nevertheless, the existence and location of the natural boundary is rather subtle, cf. e.g. the discussion in [6], and remains unknown in many seemingly simple cases. See also [1] for a survey.

1.2. **Euler products from representation growth of groups.** In this note we consider analytic properties of Euler products yet more complicated than (1.1), arising as representation zeta functions of arithmetic groups. For a group G and $n \in \mathbb{N}$ let $r_n(G)$ denote the number of inequivalent n-dimensional irreducible complex (continuous, if G is topological) representations of G. We call G (*representation*) *rigid* if $r_n(G)$ is finite for all n. In this

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case, the *representation zeta function*

$$\zeta_G(s) = \sum_{n=1}^{\infty} r_n(G) n^{-s}$$

of G converges for some $s \in \mathbb{C}$ if and only if G has *polynomial representation growth*, i.e. $r_n(G)$ is bounded by a polynomial in n. This holds for certain arithmetic groups of type A_2 defined over number fields, such as $\mathsf{SL}_3(\mathbb{Z})$. The zeta functions of these groups were intensely studied in [4].

We proceed to describe the precise set-up, recalling well-known facts, e.g. from [4, § 1.1]. Throughout this paper, let k be a number field (i.e. a finite extension of $\mathbb{Q}$) or a function field (i.e. a finite extension of $\mathbb{F}_{\mathfrak{q}}(T)$) with ring of integers $\mathcal{O}$. For a place v of k, we write k_v for the completion of k at v and, if v is non-Archimedean, $\mathcal{O}_v$ for the completion of $\mathcal{O}$ at v. Let S be a finite set of places of k, including all the Archimedean ones in the number field case, and let $\mathcal{O}_S = \{x \in k \mid x \in \mathcal{O}_v \text{ for all } v \notin S\}$ be the ring of S-integers in k.

Let $\mathbf{H}$ be a connected, simply-connected absolutely almost simple algebraic group defined over k, with a fixed embedding into GL_d for some $d \in \mathbb{N}$. We consider the arithmetic group $\mathbf{H}(\mathcal{O}_S) = \mathbf{H}(k) \cap \mathsf{GL}_d(\mathcal{O}_S)$. If $\mathbf{H}(\mathcal{O}_S)$ has the strong Congruence Subgroup Property (i.e. the congruence kernel $\ker(\widehat{\mathbf{H}(\mathcal{O}_S)} \to \mathbf{H}(\widehat{\mathcal{O}_S}))$ is trivial), abbreviated by sCSP, then we have

$$\zeta_{\mathbf{H}(\mathcal{O}_S)}(s) = \zeta_{\mathbf{H}(\mathbb{C})}(s)^r \prod_{v \notin S} \zeta_{\mathbf{H}(\mathcal{O}_v)}(s), \tag{1.2}$$

where r is the degree of k over $\mathbb{Q}$ in the number field case and 0 in the function field case; cf. [16, Proposition 1.3]. The Archimedean factors $\zeta_{\mathbf{H}(\mathbb{C})}$ enumerate the finite-dimensional, irreducible rational representations of the algebraic group $\mathbf{H}(\mathbb{C})$; their contribution to the Euler product reflects Margulis super-rigidity. The non-Archimedean Euler factors indexed by the places not in S are all rational functions, albeit not just in q_v and q_v^{-s}, where q_v denotes the residue field cardinality at v. Computing these rational functions has proven to be very challenging. Explicit formulae seem only to be known for groups of (Lie) type A_2 and A_1.

1.3. **Groups of type A_2.** Assume now that $\mathbf{H}$ is a connected, simply-connected, absolutely almost simple algebraic group of type A_2 defined over a number field k or over a function field k with characteristic greater than 3. We consider the situation when it is either an inner form arising from a matrix algebra over a central division algebra over k or an outer form over a central division algebra over a quadratic extension K/k (with the same field $\mathbb{F}_{\mathfrak{q}}$ of constants in the function field case so that K/k is a "geometric" extension).

Assume that $\mathbf{H}(\mathcal{O}_S)$ has the sCSP, so (1.2) applies. It is known that the abscissa of convergence of $\zeta_{\mathbf{H}(\mathcal{O}_S)}$ is equal to 1 and that it has meromorphic continuation to $\Re s > 1 - \delta$ for some $\delta > 0$ (specifically, $\delta = 1/6$ if k is a number field) with a double pole at $s = 1$ and no pole in $1 - \delta < \Re s < 1$; see [18, Thm. A].

In this paper we prove the following best possible refinement of these results, uniformly for number fields and function fields. Let $\zeta_3 \in \overline{k}$ denote a primitive third root of unity. If k is a number field we distinguish two cases:

$$\begin{cases} \textit{Case (A)}: & (\mathbf{H} \text{ is an outer form and } K = k(\zeta_3)) \text{ or } (\mathbf{H} \text{ is an inner form and } \zeta_3 \in k), \\ \textit{Case (B)}: & \text{otherwise.} \end{cases}$$

For the investigation of $\zeta_{\mathrm{SL}_3(\mathbb{Z})}$, for instance, Case (B) applies, since we are in the case of an inner form and $\zeta_3 \notin \mathbb{Q}$.

If k is a function field with field of constants $\mathbb{F}_{\mathfrak{q}}$, we define $\eta = 0$ if $\zeta_3 \in k$, equivalently $\mathfrak{q} \equiv 1 \pmod 3$, and $\eta = 1$ if $\zeta_3 \notin k$, equivalently $\mathfrak{q} \equiv -1 \pmod 3$. Here we distinguish the cases (A): $\mathbf{H}$ is an inner form and (B): $\mathbf{H}$ is an outer form.

Theorem 1.1. *Assume that $\mathbf{H}(\mathcal{O}_S)$ is an arithmetic group of type A_2 defined as above over a global field k with characteristic 0 or greater than 3, as described above, satisfying the sCSP. Then the function $\zeta_{\mathbf{H}(\mathcal{O}_S)}$ has meromorphic continuation to $\Re s > 5/8$ and a natural boundary at $\Re s = 5/8$.*

If k is a number field, it has a double pole at $s = 1$ and a pole at $s = 4/5$ of order 9 in Case (A) and of order 5 in Case (B), and it has no other poles in $\Re s > 3/4$.

If k is a function field with field of constants $\mathbb{F}_{\mathfrak{q}}$, it has double poles at $s \in 1 + \frac{2\pi i}{\log \mathfrak{q}}\mathbb{Z}$ and poles at $s \in 4/5 + \frac{\pi i}{\log \mathfrak{q}}(\eta + 2\mathbb{Z})$ of order 9 in Case (A) and of order 5 in Case (B), and it has no other poles in $\Re s > 3/4$.

Remark 1.2.

(1) That the degree of representation growth, i.e. the abscissa of convergence of $\zeta_{\mathbf{H}(\mathcal{O}_S)}$, only depends on the Lie type of the abstract group $\mathbf{H}$ (and not, for instance, on the ring $\mathcal{O}_S$) is an instance of a more general phenomenon: the degree of representation growth of an arithmetic group of the form $\mathbf{H}(\mathcal{O}_S)$ with the (weak) Congruence Subgroup Property (i.e. with finite congruence kernel) only depends on the root system associated with the algebraic group $\mathbf{H}$; see [3, Thm. 1.1]. As similar invariance phenomenon is established in [8] for representation zeta functions associated with unipotent group schemes.

In contrast, we see here for the first time a situation where finer invariants, such as the pole order of the second right-most pole of $\zeta_{\mathbf{H}(\mathcal{O}_S)}$, here at $s = 4/5$, depend more subtly on the underlying arithmetic structures.

(2) It would be interesting to have a conceptual explanation for the constant $5/8$. The local Euler factors come naturally as finite sums, and a trivial analysis of each summand would only lead to analytic continuation up to $\Re s > 2/3$. However, there is substantial cancellation—not algebraically, but asymptotically—which allows to continue to $\Re s > 5/8$, but not further; cf. also Remark 2.2.

The proof of Theorem 1.1 is given in Section 2. It is based on an analysis of the non-Archimedean factors in (1.2) provided in [4]. The Witten zeta function $\zeta_{\mathbf{H}(\mathbb{C})}$ poses no particular difficulty, so we turn directly to the non-Archimedean part of this Euler product. As noted above, all its factors are rational functions. We recall the explicit yet intricate formulae for almost all of these functions in Section 2.2. The product over their denominators poses no difficulty, as it yields a product of two translates of the zeta function ζ_k of k, with a few Euler factors omitted. The challenge thus lies in the analysis of the product of the numerators. This Euler product is not of the form (1.1), for which the theory of [7] would be available *mutatis mutandis*, i.e. with the product running over places $v \notin S$ and p replaced by the residue field cardinality q. Instead, we are led to consider Euler products of polynomials in q^{-s} and n_i^{-s}, for polynomial expressions n_i in q, with coefficients depending (mildly) on invariants of the arithmetic group $\mathbf{H}$ and certain congruence classes of the residue field cardinality q. The remaining "exceptional" Euler factors have little bearing on this analysis; we discuss them, together with the Archimedean factors, in Section 2.6.

The hardest part of the proof is the analysis of the natural boundary. At this point the argument in the function field case diverges from the number field case.

We are able to detect a second order term in the asymptotic formula for the Dirichlet series coefficients $r_n(\mathbf{H}(\mathcal{O}_S))$ of $\zeta_{\mathbf{H}(\mathcal{O}_S)}$. In the number field case we need to insert a smooth weight for reasons to be explained in a moment.

Theorem 1.3. *Assume that $\mathbf{H}(\mathcal{O}_S)$ is as in Theorem 1.1. There exist polynomials $P, \tilde{P} \in \mathbb{R}[X]$ with $\deg P = 1$ and $\deg \tilde{P} = 8$ in Case (A) and $\deg \tilde{P} = 4$ in Case (B), so that for every $\epsilon > 0$ the following hold:*

If k is a number field, then

$$\sum_{n=1}^{\infty} r_n(\mathbf{H}(\mathcal{O}_S))e^{-n/x} = xP(\log x) + x^{4/5}\tilde{P}(\log x) + O_\epsilon(x^{3/4+\epsilon}). \tag{1.3}$$

If k is a function field with constant field $\mathbb{F}_{\mathfrak{q}}$, then

$$r_{\mathfrak{q}^n}(\mathbf{H}(\mathcal{O}_S)) = \mathfrak{q}^n P(\log \mathfrak{q}^n) + \mathfrak{q}^{\frac{4}{5}n}\tilde{P}(\log \mathfrak{q}^n) + O_\epsilon(\mathfrak{q}^{n(3/4+\epsilon)}).$$

Remark 1.4. That we cannot expect a power-saving in a sharp cut-off count in the situation of (1.3) is not surprising, as an inspection of the Euler product over the factors (2.3) shows: it is at least as difficult as the basic function $\sum_n \phi(n)^{-s} = \zeta(s)\prod_p(1+(p-1)^{-s}-p^{-s})$ where at the current state of knowledge for the summatory function $\sum_{\phi(n)\le x} 1$ no power-saving is available; see [5].

Remark 1.5. The identity (1.3) invites a comparison with the asymptotic statement that

$$\sum_{n=1}^{x} r_n(\mathbf{H}(\mathcal{O}_S)) \sim c(\mathbf{H}(\mathcal{O}_S)) \cdot x \log x \quad \text{ for } x \to \infty \tag{1.4}$$

for a constant $c(\mathbf{H}(\mathcal{O}_S)) \in \mathbb{R}_{>0}$; see [4, Cor. B(2)]. Comparing (1.4) with (1.3) yields that $c(\mathbf{H}(\mathcal{O}_S))$ is the leading coefficient of P_1. The constant $c(\mathsf{SL}_3(\mathbb{Z}))$ is discussed in [4, Sec. 7.1].

1.4. **Groups of type A_1.** Assume now that $\mathbf{H}$ is of type A_1, viz. a form of SL_2, and that S contains all places dividing 2 and ∞. The group $\mathsf{SL}_2(\mathbb{Z})$ does not have the strong Congruence Subgroup Property, but groups of the form $\mathsf{SL}_2(\mathcal{O}_S)$ do, for sufficiently large finite sets of places S. Zeta functions of groups of the form $\mathbf{H}(\mathcal{O}_S)$ with the sCSP have been considered e.g. in [16, § 10]. By [16, Thm. 10] we know that $\zeta_{\mathbf{H}(\mathcal{O}_S)}$ has abscissa of convergence equal to 2. A porism of their result is that the Euler product (1.2) allows for some meromorphic continuation, unveiling a simple pole at $s = 2$. We extend these results as follows.

Theorem 1.6. *Assume that $\mathbf{H}(\mathcal{O}_S)$ is an arithmetic group of type A_1 defined as above, over a number field or over a function field of characteristic greater than 3, satisfying the sCSP. The function $\zeta_{\mathbf{H}(\mathcal{O}_S)}$ has meromorphic continuation to $\Re s > 1$ with a simple pole at $s = 2$ (resp. simple poles at $s = 2 + \frac{2\pi i}{\log \mathfrak{q}}n$, $n \in \mathbb{Z}$, if k is a function field with constant field $\mathbb{F}_{\mathfrak{q}}$) and no further poles. It has a branch cut singularity at $s = 1$.*

There exists a constant $c > 0$ such that for every $\epsilon > 0$ the following hold:

If k is a number field, then

$$\sum_{n=1}^{\infty} r_n(\mathbf{H}(\mathcal{O}_S))e^{-n/x} = cx^2 + O_\epsilon(x^{1+\epsilon}).$$

If k is a function field with constant field $\mathbb{F}_{\mathfrak{q}}$, then

$$r_{\mathfrak{q}^n}(\mathbf{H}(\mathcal{O}_S)) = c\mathfrak{q}^{2n} + O_\epsilon(\mathfrak{q}^{n(1+\epsilon)}).$$

In particular, there exists no $\delta > 0$ such that $\zeta_{\mathbf{H}(\mathcal{O}_S)}$ has a meromorphic extension to $\Re s > 1 - \delta$.

2. Proof of Theorem 1.1

2.1. **Preliminaries.** Recall that $\mathbf{H}$ is either an inner or an outer form. In the latter case let χ denote the quadratic character of k associated with the relevant extension K/k describing the splitting behaviour of a prime ideal $\mathfrak{p}$ of k in K. In the former case we denote by χ simply the trivial character.

If necessary, we momentarily enlarge the set S to include the finitely many finite places v of k (if any) where K/k ramifies or whose residue field cardinality is divisible by 2 or 3.

Let $v \notin S$ be a place corresponding to a prime ideal $\mathfrak{p}$ of k with residue field cardinality $q = N\mathfrak{p}$. As in [4, (1.4)] we define

$$\varepsilon = \varepsilon_v = \chi(\mathfrak{p}) \in \{-1, 1\}.$$

If $\varepsilon = 1$, then $\mathbf{H}(\mathcal{O}_v) \cong \mathrm{SL}_3(\mathcal{O}_v)$; if $\varepsilon = -1$, then $\mathbf{H}(\mathcal{O}_v) \cong \mathrm{SU}_3(\mathcal{O}_v)$.

Let ψ denote the character defined by

$$\psi_v = \psi(\mathfrak{p}) = \Big(\frac{-3}{q}\Big) = \begin{cases} 1, & q \equiv 1 \,(\mathrm{mod}\ 3), \\ -1, & q \equiv -1 \,(\mathrm{mod}\ 3). \end{cases}$$

As in [4, (1.10)] we define

$$\iota = \iota(\varepsilon, q) = (q - \varepsilon, 3) = 2 + \epsilon_v \psi_v \in \{1, 3\}.$$

Note that ε and hence ι depend on v, not only on q.

We write $\zeta_k^S(s) = \prod_{v \notin S} \zeta_{k,v}(s)$ for the zeta function $\zeta_k(s) = \prod_v \zeta_{k,v}(s)$ without the local factors indexed by places $v \in S$, and similarly for the L-function $L^S(s, \chi\psi)$.

In the function field case, the zeta- and L-functions under consideration are periodic with respect to $s \mapsto s + \frac{2\pi i}{\log \mathfrak{q}}$, since its Dirichlet coefficients are indexed only by powers of $\mathfrak{q}$. We note that ζ_k has a simple poles at $s = 0$ and $s = 1$ (resp. $j + \frac{2\pi i}{\log \mathfrak{q}} n$, $n \in \mathbb{N}$, $j \in \{0, 1\}$ in the function field case), but is holomorphic otherwise. We need to understand the analytic behaviour of $L(., \chi\psi)$.

We first assume that k is a number field. Then χ and ψ are Hecke characters, and we observe that $\chi\psi$ is trivial if and only we are in Case (A). Indeed, ψ is trivial if and only if $\zeta_3 \in k$. If ψ is non-trivial, then K/k is uniquely determined by the condition $\chi = \psi$, and we see that for $K = k(\zeta_3)$ the character of the extension K/k equals ψ.

We now assume that k is a function field. Then χ is trivial if and only if we are in case (A), namely $\mathbf{H}$ in an inner form. Moreover, ψ is the again the character associated with the extension $k(\zeta_3)/k$, which is either trivial (if $\zeta_3 \in k$, equivalently $\eta = 0$) or a quadratic constant field extension (if $\zeta_3 \notin k$, equivalently $\eta = 1$). From [10, Prop. 5.3.2] we conclude that

$$L(s, \chi\psi) = L\Big(s + \eta \frac{\pi}{\log \mathfrak{q}}, \chi\Big) \tag{2.1}$$

which is entire if and only if χ is trivial (since K/k is not a constant field extension).

2.2. **Generic Euler factors.** Our starting point for the proof of Theorem 1.1 is the Euler product (1.2). We use the explicit formula [4, Corollary D], cf. [18, Thm. C] in the function field case, and consider the product

$$\prod_{v\notin S}\left(\zeta_{\mathbf{H}(\mathbb{F}_q)}(s)+\psi_{\varepsilon,q}(s)\right), \tag{2.2}$$

where

$$\begin{aligned}\zeta_{\mathbf{H}(\mathbb{F}_q)}(s) =& 1+\frac{1}{(q^2+\varepsilon q)^s}+\frac{q-1-\varepsilon}{(q^2+\varepsilon q+1)^s}+\frac{q^2-q-1+\varepsilon}{2(q^3-\varepsilon)^s}+\frac{1}{q^{3s}}+\frac{q-1-\varepsilon}{(q^3+\varepsilon q^2+q)^s}\\ &+\frac{q^2+\varepsilon q-2+2\iota(\varepsilon,q)^{2+s}}{3((q+\varepsilon)(q-\varepsilon)^2)^s}+\frac{(q-\varepsilon)(q-3-\varepsilon)+2\iota(\varepsilon,q)^{2+s}}{6((q^2+\varepsilon q+1)(q+\varepsilon))^s}\end{aligned} \tag{2.3}$$

is the representation zeta function of the finite group of Lie type $\mathbf{H}(\mathbb{F}_q)$ and

$$\begin{aligned}\psi_{\varepsilon,q}(s) = \Big(&\frac{(1-q^{2-3s})(q-1)(q-\varepsilon)(2+2q^{-s}+(q-2)(q+1)^{-s}+q(q-1)^{-s})}{2(q^2(q^2+\varepsilon q+1))^s}\\ &+\frac{(1-q^{2-3s})(q-\varepsilon+\iota(\varepsilon,q)^{2+s}(q+\varepsilon)(q-\varepsilon)^{-s}+\iota(\varepsilon,q)^2(q-1)(q^2-1)q^{-s})}{((q^3-\varepsilon)(q+\varepsilon))^s}\\ &+\frac{(q-1)(q-\varepsilon)^2(q-2+2q^{2-2s}-q^{1-2s})}{6(q^3(q^2+\varepsilon q+1)(q+\varepsilon))^s}+\frac{(q-1)(q^2-1)q(1-q^{-2s})}{2(q^3(q^3-\varepsilon))^s}\\ &+\frac{(1-q^{1-2s})(q^2-1)(q^2+\varepsilon q+1)}{3(q^3(q^2-1)(q-\varepsilon))^s}+\frac{(q-1)(q-\varepsilon)q(1+q^{1-2s})}{(q^2(q^3-\varepsilon)(q+\varepsilon))^s}\\ &+\frac{(1-q^{-2s})q^2\iota(\varepsilon,q)^{2+s}}{(q(q^3-\varepsilon)(q^2-1))^s}+\frac{(\varepsilon+1)\iota(\varepsilon,q)^{2+s}q^{2-2s}}{((q^3-1)(q^2-1)q)^s}\Big)\frac{1}{(1-q^{1-2s})(1-q^{2-3s})}.\end{aligned} \tag{2.4}$$

We note that each of these Euler factors is a rational function in finitely many numbers n_i^{-s} (with n_i depending on q, ε, and ι) and hence a meromorphic function. In $\Re s > 1/2$ the only possible poles appear on the line $\Re s = 2/3$. Since $\zeta_{\mathbf{H}(\mathcal{O}_S)}$ is a generating series of non-negative objects, it is non-zero on the segment $s > 2/3$.

In order to make these terms resemble (1.1) more closely, we insert a Taylor expansion:

$$\begin{aligned}&(q+\varepsilon)^{-s} = q^{-s}\Big(1-\frac{\varepsilon s}{q}+O\Big(\frac{|s|^2}{q^2}\Big)\Big),\\ &(q^2+\varepsilon q+1)^{-s} = q^{-2s}\Big(1-\frac{\varepsilon s}{q}+O\Big(\frac{|s|^2}{q^2}\Big)\Big),\\ &(q^3-\varepsilon)^{-s} = q^{-3s}\Big(1+O\Big(\frac{|s|^2}{q^2}\Big)\Big),\\ &((q+\varepsilon)(q-\varepsilon)^2)^{-s} = q^{-3s}\Big(1+\frac{\varepsilon s}{q}+O\Big(\frac{|s|^2}{q^2}\Big)\Big),\\ &((q^2+\varepsilon q+1)(q\pm 1))^{-s} = q^{-3s}\Big(1-\frac{(\varepsilon\pm 1)s}{q}+O\Big(\frac{|s|^2}{q^2}\Big)\Big),\\ &((q^3-\varepsilon)(q^2-1))^{-s} = q^{-5s}\Big(1+O\Big(\frac{|s|^2}{q^2}\Big)\Big),\end{aligned} \tag{2.5}$$

$$((q^2-1)(q-\varepsilon))^{-s} = q^{-3s}\Big(1+\frac{\varepsilon s}{q}+O\Big(\frac{|s|^2}{q^2}\Big)\Big),$$
$$((q^3-\varepsilon)(q+\varepsilon))^{-s} = q^{-4s}\Big(1-\frac{\varepsilon s}{q}+O\Big(\frac{|s|^2}{q^2}\Big)\Big).$$

Plugging this into the previous equation, we obtain after a straightforward computation

$$\begin{aligned}\mathcal{E}(s,q) &:= (1-q^{1-2s})(1-q^{2-3s})\left(\zeta_{\mathbf{H}(\mathbb{F}_q)}(s)+\psi_{\varepsilon,q}(s)\right)\\ &= 1+\iota(\varepsilon,q)^2(q^{3-5s}-q^{5-8s})+F(s,q),\end{aligned} \tag{2.6}$$

say, with

$$F(s,q) \ll_s q^{4-8\Re s}+q^{2-5\Re s}+q^{-2\Re s} \ll q^{4-8\Re s}+q^{-2\Re s} \tag{2.7}$$

and

$$\frac{d}{ds}F(s,q) = \int_{|z-s|=(\log q)^{-1}} \frac{F(z,q)}{z-s}\frac{dz}{2\pi i} \ll_s (q^{4-8\Re s}+q^{-2\Re s})\log q.$$

Remark 2.1. The summands in (2.4) reflect the organization of the relevant characters according to invariants called *shadows* in [4]. Our analysis shows that no single shadow or summand in (2.4) suffices to explain the asymptotic properties established in Theorem 1.1. The relatively simple shape of (2.6) is therefore quite remarkable: both (2.3) and (2.4) contribute additional terms of the form q^{4-6s}, but they cancel. If the did not, we could only continue to $\Re s > 2/3$, cf. also Remark 2.2.

In the following three sections we show that the infinite product (2.2) has meromorphic continuation to $\Re s > 5/8$ and a natural boundary at $\Re s = 5/8$. In Section 2.6 we show that these properties remain true after adding the finitely many missing Euler factors.

2.3. **The analytic continuation.** We now start with the proof of the analytic continuation to $\Re s > 5/8$. For $n, m \in \mathbb{Z}$ with $n \equiv m \pmod 2$ we define the polynomials

$$P_+(x;n,m) = \begin{cases}(1-x)^n, & |m|\le n,\\ (1+x)^{-(n+m)/2}(1-x)^{(n-m)/2}, & |n|\le -m,\\ (1+x)^{-n}, & |m|\le -n,\\ (1-x)^{(n+m)/2}(1+x)^{-(n-m)/2}, & |n|\le m,\end{cases}$$

$$P_-(x;n,m) = \begin{cases}(1-x)^{(n+m)/2}(1+x)^{(n-m)/2}, & |m|\ge n,\\ (1+x)^{-m}, & |n|\le -m\\ (1+x)^{-(n+m)/2}(1-x)^{-(n-m)/2}, & |m|\le -n,\\ (1-x)^m, & |n|\le m.\end{cases}$$

Clearly we have $P_+(x;n,m) \equiv P_-(x;n,m) \pmod 2$ and also

$$P_+(x;n,m) = 1-nx+\dots,\quad P_-(x;n,m) = 1-mx+\dots. \tag{2.8}$$

By direct comparison of Euler products we obtain

$$\prod_{\substack{v\notin S\\ \varepsilon_v\psi_v=1}} P_+(q^{-s};n,m) \prod_{\substack{v\notin S\\ \varepsilon_v\psi_v=-1}} P_-(q^{-s};n,m)$$

$$= \zeta_k^S(s)^{-\frac{n+m}{2}} L^S(s,\chi\psi)^{-\frac{n-m}{2}} \cdot \begin{cases} 1, & |m|\leq n,\\ \zeta_k^S(2s)^{\frac{n+m}{2}}, & |n|\leq -m,\\ \zeta_k^S(2s)^n, & |m|\leq -n,\\ \zeta_k^S(2s)^{\frac{n-m}{2}}. & |n|\leq m.\end{cases} \tag{2.9}$$

We now return to the right hand side of (2.6) and modify the ideas of [7, Lemma 5.5]. We put $X=q$, $Y=q^{-s}$ and consider the polynomials

$$W_{+,0}(X,Y)=1+9X^3Y^5-9X^5Y^8,\quad W_{-,0}(X,Y)=1+X^3Y^5-X^5Y^8.$$

We proceed recursively as follows: given two polynomials

$$W_{+,j}=\sum_{m,n}\alpha_{m,n}^{(j)}X^nY^m,\quad W_{-,j}=\sum_{m,n}\beta_{m,n}^{(j)}X^nY^m\in\mathbb{Z}[X,Y]$$

with $\alpha_{m,n}^{(j)}\equiv\beta_{m,n}^{(j)} \pmod 2$ and $\alpha_{0,0}^{(j)}=\beta_{0,0}^{(j)}=1$, we order the monomials lexicographically by their exponents (m,n) and pick the smallest index $(m_j,n_j)>(0,0)$ such that $\alpha_{m_j,n_j}^{(j)}$ or $\beta_{m_j,n_j}^{(j)}$ are non-zero. We define

$$W_{\pm,j+1}(X,Y)=W_{\pm,j}(X,Y)P_\pm(X^{n_j}Y^{m_j};\alpha_{m_j,n_j}^{(j)},\beta_{m_j,n_j}^{(j)}).$$

Then the polynomials $W_{\pm,j+1}$ have again constant term 1 and are congruent modulo 2. Moreover, the smallest nontrivial monomial $X^{n_j}Y^{m_j}$ is cleared by (2.8) and no smaller monomials are inferred. Finally, inductively we see easily that only monomials of the form

$$(X^3Y^5)^u(X^5Y^8)^v=X^{3u+5v}Y^{5u+8v} \tag{2.10}$$

for $u,v\in\mathbb{N}_0$ can occur in $W_{\pm,j}$.

For illustration we carry out the first two steps of this procedure. We have $(m_0,n_0)=(5,3)$, $\alpha_{5,3}^{(0)}=9$, $\beta_{5,3}^{(0)}=1$ and

$$P_\pm(X^3Y^5,9,1)=\begin{cases}(1-X^3Y^5)^9, & \pm=+,\\ (1-X^3Y^5)^5(1-X^3Y^5)^4, & \pm=-,\end{cases} \tag{2.11}$$

and so

$$W_{\pm,1}=\begin{cases}1-9X^5Y^8-45X^6Y^{10}+\ldots+9X^{32}Y^{53}, & \pm=+,\\ 1-X^5Y^8-5X^6Y^{10}+\ldots+X^{32}Y^{53}, & \pm=-.\end{cases} \tag{2.12}$$

Next, we have $(m_1,n_1)=(8,5)$ and $\alpha_{8,5}^{(1)}=-9$, $\beta_{8,3}^{(1)}=-1$ and

$$P_\pm(X^5Y^8,-9,-1)=\begin{cases}(1+X^5Y^8)^9, & \pm=+,\\ (1+X^5Y^8)^5(1+X^5Y^8)^4, & \pm=-,\end{cases}$$

getting

$$W_{\pm,2}=\begin{cases}1-45X^6Y^{10}+\ldots+9X^{77}Y^{125}, & \pm=+,\\ 1-5X^6Y^{10}+\ldots+X^{77}Y^{125}, & \pm=-.\end{cases}$$

For $v \notin S$ and $i \in \mathbb{N}$ let us define

$$\mathcal{P}_{i,v} := \prod_{j=0}^{i-1} P_{\varepsilon_v \psi_v}(q^{n_j + s m_j}; \alpha^{(j)}_{m_j,n_j}, \beta^{(j)}_{m_j,n_j}).$$

Clearly, we have

$$W_{\pm,0}(X,Y) = W_{\pm,i}(X,Y) \prod_{j=0}^{i-1} P_{\pm}(X^{n_j} Y^{m_j}; \alpha^{(j)}_{m_j,n_j}, \beta^{(j)}_{m_j,n_j})^{-1}$$

for any $i \in \mathbb{N}$, and by (2.9) and (2.10) we know that

$$\prod_{\pm} \prod_{\substack{v \notin S \\ \varepsilon_v \psi_v = \pm 1}} \mathcal{P}_{i,v}^{-1}$$

is a finite product of positive or negative integral powers of

$$\zeta_k^S((5u+8v)s - (3u+5v)) \quad \text{and} \quad L^S((5u+8v)s - (3u+5v), \chi\psi) \tag{2.13}$$

for certain $u, v \in \mathbb{N}_0$, $(u,v) \neq (0,0)$, in particular meromorphic in s.

Returning to (2.6), we can write

$$\mathcal{E}(s,q) = W_{\varepsilon_v\psi_v,0}(q, q^{-s}) + F(s,q) = \big(W_{\varepsilon_v\psi_v,i}(q,q^{-s}) + F(s,q)\mathcal{P}_{i,v}\big)\mathcal{P}_{i,v}^{-1} \tag{2.14}$$

for any $i \in \mathbb{N}$. Here

$$W_{\varepsilon_v\psi_v,i}(q,q^{-s}) = 1 + q^{(3u_0+5v_0)-(5u_0+8v_0)s} + \dots$$

where (u_0, v_0) can be chosen as large as we wish by choosing i sufficiently large. Moreover, by (2.7) we see that $F(s,q)\mathcal{P}_{i,v}$ is bounded by

$$q^{(4+3u+5v)-(8+5u+8v)\Re s} + q^{(3u+5v)-(8+5u+8v+2)\Re s}$$

for certain positive integers u, v. We conclude that

$$\prod_{\pm} \prod_{\substack{v \notin S \\ \varepsilon_v \psi_v = \pm 1}} \big(W_{\varepsilon_v\psi_v,i}(q,q^{-s}) + F(s,q)\mathcal{P}_{i,v}\big) \tag{2.15}$$

is absolutely convergent in $\Re s \geq 5/8 + \delta$ for any $\delta > 0$. In this way we obtain a meromorphic continuation of

$$\zeta_{\mathbf{H}(\mathcal{O}_S)}(s) = \zeta_k^S(2s-1)\zeta_k^S(3s-2) \prod_{v \notin S} \mathcal{E}(s,q)$$

to the half plane $\Re s > 5/8$.

Carrying out only the first step of the inductive procedure described above, we see from (2.11) and (2.12) that

$$\zeta_{\mathbf{H}(\mathcal{O}_S)}(s) = \zeta_k^S(2s-1)\zeta_k^S(3s-2)\zeta_k^S(5s-3)^5 L^S(5s-3,\chi\psi)^4 H(s) \tag{2.16}$$

where H is an absolutely convergent Euler product in $\Re s > 3/4$ and hence in particular holomorphic and non-vanishing.

Thus in $\Re s > 3/4$ the function $\zeta_{\mathbf{H}(\mathcal{O}_S)}$ has the polar behaviour described in Theorem 1.1.

2.4. **The natural boundary – number field case.** Next we show that the line $\Re s = 5/8$ is a natural boundary. To this end, we show that every given point $s_0 = 5/8 + it$ is a limit point of zeros. For clarity we assume in this subsection that k is a number field and explain the necessary modifications in the function field case in the next subsection. In both cases we show, roughly speaking, that sufficiently many Euler factors have zeros in a small neighbourhood of s_0. A difficulty is, however, to make sure that these zeros are not cancelled by poles for the zeta- and L-functions that arise in the course of the analytic continuation. Here the argument diverges in the number and function field case. In the former we use bounds for the total number of zeros of zeta functions over number fields. In the latter we use the Riemann hypothesis (which is known over function fields) along with Kronecker's simultaneous approximation theorem to compensate for the fact that we have only very few different norms at our disposal (namely those that are powers of $\mathfrak{q}$).

We proceed to show that $\Re s = 5/8$ is a natural boundary in the number field case. Let $c_0 = c_0(t) = 9(\pi + |t| \log 2)/\log 9$. Fix some small $\delta > 0$ and consider the rectangle

$$\mathcal{R}_\delta = \{s \in \mathbb{C} : |\Re(s - s_0 - \delta)| \leq \delta/2, |\Im(s - s_0)| \leq c_0\delta\}$$

with a typical Euler factor $\mathcal{E}(s, q)$ as in (2.6) corresponding to a place v with $\varepsilon_v \psi_v = 1$. We recall that in this case

$$\mathcal{E}(s, q) = 1 + 9q^{3-5s} - 9q^{5-8s} + O(q^{4-8\Re s} + q^{2-\Re s}). \tag{2.17}$$

In what follows we always assume that q is sufficiently large in terms of t and the fixed field extension K/k. We will choose later q to be a function of δ and let δ tend to zero.

For $n \in \mathbb{Z}$ we are looking for zeros of $\mathcal{E}(s, q)$ in a neighbourhood of the points

$$s = \frac{5}{8} + \frac{i(1 + 2n)\pi}{\log q}.$$

A good approximation can be found by putting $V = q^{-1/8}$, $U = q^{5/8-s}$ and writing down the Puiseux series in V of the equation $1 + 9U^5V - 9U^8 = 0$ near $U = -3^{-1/4}$. One checks that

$$1 + 9q^{3-5s} - 9q^{5-8s}\big|_{s=\frac{5}{8}+\frac{1}{\log q}(i(1+2n)\pi+\frac{\log 9}{8})} \ll q^{-1/8}$$

and hence there must be constants c_j such that for

$$s_{n,q} := \frac{5}{8} + \frac{1}{\log q}\Big(i(1 + 2n)\pi + \frac{\log 9}{8} + \sum_{j=1}^{7} c_j q^{-j/8}\Big) \tag{2.18}$$

we have

$$1 + 9q^{3-5s_{n,q}} - 9q^{5-8s_{n,q}} \ll q^{-1}.$$

(While not relevant for the following discussion, the constants are

$$c_1 = \frac{3^{3/4}}{8}, \quad c_2 = -\frac{3\sqrt{3}}{64}, \quad c_3 = -\frac{21 \cdot 3^{1/4}}{1024}, \quad c_4 = \frac{27}{512}, \quad c_5 = -\frac{9639 \cdot 3^{3/4}}{1310720},$$
$$c_6 = -\frac{2079\sqrt{3}}{131072}, \quad c_7 = \frac{5942079 \cdot 3^{1/4}}{234881024}.)$$

Note that $\Re s_{n,q} > 5/8$. We assume $n \ll \log q$, so that $s_{n,q} \ll 1$. By (2.17) we obtain

$$\mathcal{E}(s_{n,q}, q) \ll q^{-1}.$$

On the other hand, for $|s - s_{n,q}| \ll q^{-1}$ we compute

$$\frac{d}{ds}\mathcal{E}(s,q) = (8q^{5-8s} - 5q^{3-5s})\log q + \frac{d}{ds}F(s,q) = 8 + O(q^{-1/8}\log q).$$

Thus for q sufficiently large and $n \ll \log q$ we find a point $s^*_{n,q}$ (for instance by Newton's method) with

$$s^*_{n,q} - s_{n,q} \ll q^{-1} \tag{2.19}$$

such that $\mathcal{E}(s^*_{n,q}, q) = 0$. Note that for $q \leq q' < 2q$ we have

$$\begin{aligned} |s^*_{n,q} - s^*_{n,q'}| \geq |\Im(s^*_{n,q} - s^*_{n,q'})| &= (1+2n)\pi\Big(\frac{1}{\log q} - \frac{1}{\log q'}\Big) + O\Big(\frac{1}{q}\Big) \\ &\asymp \frac{|1+2n|(q'-q)}{q(\log q)^2} + O\Big(\frac{1}{q}\Big) > 0 \end{aligned} \tag{2.20}$$

provided that $q' - q \geq (\log q)^3$. (Note that because of the error term $F(s,q)$ we cannot use algebraic arguments such as [7, p. 130 or p. 133].)

We now define q_0 and n by

$$\delta = \frac{\log 9}{8\log q_0}, \quad n = \Big[\frac{t\log q_0}{2\pi}\Big] \tag{2.21}$$

and restrict to numbers $q \in \mathcal{I} = [q_0, 2q_0]$. Then for $q \in \mathcal{I}$ we have

$$\Big|\frac{(1+2n)\pi}{\log q} - t\Big| \leq \frac{\pi + |t|\log 2}{\log q_0} = \frac{8}{9}c_0\delta$$

and we conclude from (2.19) that for sufficiently large q we have

$$|\Im s^*_{n,q} - t| \leq c_0\delta, \quad \Big|\Re s^*_{n,q} - \frac{5}{8} - \delta\Big| \ll \delta^2 \leq \frac{1}{2}\delta$$

and so

$$s^*_{n,q} \in \mathcal{R}_\delta.$$

By (2.20) all $s^*_{n,q}$ with $q \in \mathcal{I}$ are pairwise distinct, provided the values of q are at least $(\log q_0)^3$-spaced. We can force this by restricting to places v such that $q \equiv 1 \pmod{\lceil(\log q_0)^3\rceil}$. We recall in addition the condition $\epsilon_v\psi_v = 1$, which is another set of (fixed) congruences modulo $3 \cdot \mathrm{Nr}(\Delta)$ where $\Delta \in k$ is the discriminant of the extension K/k. The number of such places can be evaluated by a number field version of the Siegel-Walfisz theorem [17] with the required uniformity in the modulus: we find

$$\asymp_{K/k} \frac{q_0}{\log q_0} \cdot \frac{1}{\log^{3[k:\mathbb{Q}]} q_0} \gg q_0^{0.9} \geq \exp(\tfrac{1}{10}\delta^{-1})$$

places v with $q \in \mathcal{I}$ such that $\mathcal{E}(s,q)$ has a zero in $\mathcal{R}_\delta$.

We now return to (2.14) and write

$$\mathcal{E}(s,q) = \big(\mathcal{E}(s,q)\mathcal{P}_{i,v}\big) \cdot \mathcal{P}_{i,v}^{-1}$$

for $i \in \mathbb{N}$. By the discussion around (2.15), the global Euler product $\prod_{v \notin S}\mathcal{E}(s,q)\mathcal{P}_{i,v}$ is absolutely convergent in $\mathcal{R}_\delta$ upon choosing i sufficiently large in terms of δ, and we know that at least $\exp(\frac{1}{10}\delta^{-1})$ Euler factors corresponding to places v with $q \in \mathcal{I}$ have a zero $\rho \in \mathcal{R}_\delta$, and all these zeros are distinct.

On the other hand, we can express the global Euler product $\prod_{v \notin S}\mathcal{P}_{i,v}^{-1}$ as a finite product of positive or negative integral powers of zeta- and L-functions of the form (2.13). Suppose that some of the above zeros $\rho \in \mathcal{R}_\delta$ coincides with a zero of some ζ- or L-factors in (2.13) (which may appear in the denominator with some multiplicity and hence cancel ρ). Then

$$\frac{5}{8} + \delta + O(\delta^2) = \Re(\rho) \leq \frac{1+3u+5v}{5u+8v} \leq \frac{1}{5u+8v} + \frac{5}{8}$$

and $\Im\rho \ll 1$, so that

$$(5u+8v)\rho - (3u+5v)$$

is a zero of ζ_k or $L(., \chi\psi)$ with imaginary part $\ll (5u+8v) \ll 1/\delta$. There are at most $O(\delta^{-1}|\log\delta|)$ such zeros (cf. [13, Thm 5.8]), and the number of choices for the pair (u,v) is $O(\delta^{-2})$, so that in total the meromorphic continuation of the global Euler product $\prod_{v\notin S} \mathcal{P}_{i,v}^{-1}$ can have at most $O(\delta^{-3}|\log\delta|)$ poles in $\mathcal{R}_\delta$. Choosing δ sufficiently small, this cannot compensate the zeros found above.

We conclude that we find a sequence of zeros of the product (2.2) converging to our given point s_0.

2.5. **The natural boundary – function field case.** The main difference in the function field case is that we have only a very sparse set of distinct values q available, namely only numbers of the form $\mathfrak{q}^m$, $m \in \mathbb{N}$. On the other hand, the Riemann hypothesis is known, which we will use in the subsequent analysis.

As before, let us fix some $s_0 = 5/8 + it$ and choose some place v with $q = \mathfrak{q}^m$ and $\varepsilon_v\psi_v = 1$. At least if m is even and sufficiently large, this is always possible: for even m we have $\psi_v = 1$ and by a basic form of the Chebotarev density theorem (e.g. [10, Proposition 7.4.8] with $n = 1$, $m = 2$) places v with $\epsilon_v = 1$ exist in abundance. Let us choose one such v for each such $q = \mathfrak{q}^{2m}$. Choose $s_{n,q}$ as in (2.18), $s^*_{n,q}$ as in (2.19) and n as in (2.21), so that $\mathcal{E}(s^*_{n,q}, q) = 0$ and

$$|\Im s^*_{n,q} - t| \ll \frac{1}{\log q}, \quad \Re s^*_{n,q} - \frac{5}{8} =: \eta_q = \frac{1}{\log q}\Big(\frac{\log 9}{8} + O(q^{-1/4})\Big). \tag{2.22}$$

For $q = \mathfrak{q}^{2m}$ this gives a sequences of distinct zeros tending to s_0. We need to show that it contains a subsequence that is not cancelled by possible zeros of the zeta- and L-factors at arguments of the form $(5u+8v)s - (3u+5v)$ with $u, v \in \mathbb{N}_0$. The Riemann hypothesis (see e.g. [10, Thm 5.5.1]) states that the zeros of $\zeta_K = \zeta_k L(., \chi)$ and hence of both ζ_k and $L(., \chi\psi)$ (cf. (2.1)) are on the line $\Re s = 1/2$. Therefore, the real parts of the zeros in question are at

$$\frac{\frac{1}{2} + 3u + 5v}{5u+8v}.$$

Elementary algebra shows that this is larger than $5/8$ only for $u < 4$, and so it can only coincide with $5/8 + \eta_q$ if (u,v) equals

$$\Big(0, \frac{1}{16\eta_q}\Big), \quad \Big(1, \frac{3}{64\eta_q} - \frac{5}{8}\Big), \quad \Big(2, \frac{1}{32\eta_q} - \frac{5}{4}\Big), \quad \text{or} \quad \Big(3, \frac{1}{64\eta_q} - \frac{15}{8}\Big).$$

In order to derive at a contradiction, we show that for an infinite subsequence of $q = \mathfrak{q}^{2m}$ the second entry is not an integer, say its fractional part $\|.\|$ is in $(1/4, 3/4)$. To this end let $\alpha := \frac{\log \mathfrak{q}}{2\log 9}$ and consider the linear polynomials

$$\ell_1(m) = 2\alpha m + \frac{1}{2}, \quad \ell_2(m) = \frac{3}{2}\alpha m - \frac{1}{8}, \quad \ell_3(m) = \alpha m - \frac{3}{4}, \quad \ell_4(m) = \frac{1}{2}\alpha m - \frac{11}{8},$$

which by the definition (2.22) of η_q describe the possible values of $v+1/2$ up to an error of size $O(\mathfrak{q}^{-2m/4})$. Since α is irrational (since 3 and $\mathfrak{q}$ are coprime), it follows from Kronecker's approximation theorem (see e.g. [12, Thm. 442]) that there are infinitely many m such that simultaneously $\|\ell_j(m)\| < 1/4$ for $1 \le j \le 4$, which by the above discussion provides the desired sequence of zeros tending to s_0.

2.6. **The remaining Euler factors.** Recall that at the beginning of the proof we have possibly enlarged the set S by finitely many places. We now deal with the local factors at these places and the infinite Euler factor. By [14, Thm. 1.1] and [2, Thm. B] the associated local representation zeta functions $\zeta_{\mathbf{H}(\mathcal{O}_v)}(s)$ are rational functions (hence meromorphic on the whole complex plane) with their right-most poles at $\Re s = 2/3$. As generating function of non-negative numbers, they are non-vanishing on the segment $s > 2/3$, so they do not change the polar behaviour there, nor do they change the natural boundary.

We finally discuss the Archimedean Euler factors of (1.2), which only occur in the number field case. They are given (cf. [15, p. 359]) by powers of

$$\zeta_{\mathrm{SL}_3(\mathbb{C})}(s) = \sum_{m,n} \frac{1}{m^s n^s (m+n)^s}.$$

It suffices to obtain meromorphic continuation past the region $\Re s > 2/3$ of absolute convergence. This is straightforward (and well-known): by the integral formula for the beta function [11, 3.196.2 with $u = 0$] we have, initially for $\Re s > 2/3$, the absolutely convergent expression

$$\begin{aligned}\zeta_{\mathrm{SL}_3(\mathbb{C})}(s) = \sum_{n,m} \frac{1}{m^{2s} n^s (1+n/m)^s} &= \sum_{m,n} \frac{1}{m^{2s} n^s} \int_{(1/3)} \frac{\Gamma(s-t)\Gamma(t)}{\Gamma(s)} \Big(\frac{n}{m}\Big)^{-t} \frac{dt}{2\pi i} \\ &= \int_{(1/3)} \frac{\Gamma(s-t)\Gamma(t)}{\Gamma(s)} \zeta(s+t)\zeta(2s-t) \frac{dt}{2\pi i}.\end{aligned}$$

Let us assume $2/3 < \Re s < 1$ and fix $\varepsilon > 0$ sufficiently small. Then shifting the contour to $\Re t = \varepsilon$, we obtain

$$\zeta_{\mathrm{SL}_3(\mathbb{C})}(s) = \int_{(\varepsilon)} \frac{\Gamma(s-t)\Gamma(t)}{\Gamma(s)} \zeta(s+t)\zeta(2s-t) \frac{dt}{2\pi i} + \frac{\Gamma(1-s)\Gamma(2s-1)}{\Gamma(s)} \zeta(3s-1).$$

This is meromorphic in $1/2 < \Re s < 1$ with a pole only at $s = 2/3$.

Remark 2.2. One can form the global Euler product

$$\prod_{v \notin S} \zeta_{\mathbf{H}(\mathbb{F}_q)}(s),$$

and by a similar Taylor expansion obtain

$$\zeta_{\mathbf{H}(\mathbb{F}_q)}(s) = 1 + q^{2-3s} + q^{1-2s} + F^*(s,q)$$

with $F^*(s,q) \ll q^{-2\Re s}$. Arguing similarly as before, we can extend the global zeta function meromorphically only to $\Re s > 2/3$. It is interesting to note that this half plane of continuation is smaller than the half plane of continuation for $\zeta_{\mathbf{H}(\mathcal{O}_S)}$. In other words, as observed before, there is some non-trivial cancellation in the local factors $\zeta_{\mathbf{H}(\mathbb{F}_q)}(s)$ and $\psi_{\varepsilon,q}(s)$.

3. Proof of Theorem 1.3

This is a standard application of Mellin inversion. The technical difficulty lies in the fact that it is not clear (and quite possibly not true) that in the number field case $\zeta_{\mathbf{H}(\mathcal{O}_S)}$ has polynomial growth on vertical lines beyond the region of absolute convergence, cf. the remark after Theorem 1.3. This is irrelevant in the function field case since all zeta functions are $\frac{2\pi i}{\log \mathfrak{q}}\mathbb{Z}$-periodic.

We first give the argument in the number field case. To obtain subexponential growth in the region $\Re s \geq 3/4$, we replace all parentheses on the right hand sides of (2.5) by

$$1 + O\Big(\min\Big(1, \frac{|s|}{q}\Big)\Big) = 1 + O\Big(\frac{|s|^{99/100}}{q^{99/100}}\Big).$$

Then by the same computation we obtain

$$F(s,q) \ll |s|^{99/100}\big(q^{1-3\Re s} + q^{-2\Re s}\big)$$

in $\Re s \geq 3/4$ where now the implied constant is absolute. For fixed $0 < \epsilon < 1/100$, we can then write (2.16) as

$$\zeta_{\mathbf{H}(\mathcal{O}_S)}(s) = \zeta_k^S(2s-1)\zeta_k^S(3s-2)\zeta_k^S(5s-3)^5 L^{(S)}(5s-3,\chi\psi)^4 H(s)$$

where

$$H(s) = H_0(s)\prod_{v\notin S}\Big(1 + \frac{O(1)}{q^{1+\epsilon}} + \frac{O(|s|^{99/100})}{q^{5/4}}\Big) \ll \exp(|s|^{99/100})$$

in $\Re s \geq 3/4+\epsilon$ where H_0 contains potentially finitely many Euler factors that we discard at the beginning of Section 2.1. By Mellin inversion (cf. [13, (4.107)]) we now obtain

$$\sum_n r_n(\mathbf{H}(\mathcal{O}_S))e^{-n/x} = \int_{(2)} \zeta_{\mathbf{H}(\mathcal{O}_S)}(s)\Gamma(s)x^s\frac{ds}{2\pi i}.$$

We shift the contour to $\Re s = 3/4+\epsilon$. The residues at $s=1$ and $s=4/5$ yield the two main terms, and the remaining integral is rapidly convergent on the line $\Re s = 3/4+\epsilon$ in view of the estimate $\Gamma(s) \ll (1+|s|)^{1/2}\exp(-\pi|s|/2)$. This completes the proof.

If k is a function field with constant field $\mathbb{F}_{\mathfrak{q}}$, then we have the Mellin formula

$$r_{\mathfrak{q}^n}(\mathbf{H}(\mathcal{O}_S)) = \int_{2-\frac{\pi i}{\log \mathfrak{q}}}^{2+\frac{\pi i}{\log \mathfrak{q}}} \zeta_{\mathbf{H}(\mathcal{O}_S)}(s)(\mathfrak{q}^n)^s(\log \mathfrak{q})\frac{ds}{2\pi i}$$

which in this case is nothing but Cauchy's integral formula for the power series $\sum_n r_{\mathfrak{q}^n}(\mathbf{H}(\mathcal{O}_S))x^n$ with $x = \mathfrak{q}^{-s}$. We can now shift the contour to $\Re s \geq 3/4+\epsilon$ in the same way as before, noting that the contribution of the horizontal lines cancel by periodicity.

4. Groups of type A_1 – proof of Theorem 1.6

This follows from the local formulae [4, (A.5)], see also [14, Thm. 7.5], which read

$$\begin{aligned}\zeta_{\mathbf{H}(\mathcal{O}_v)}(s) =& 1 + q^{-s} + \frac{(q-3)}{2(q+1)^s} + \frac{(q-1)^{1-s}}{2} + 2\Big(\frac{q+1}{2}\Big)^{-s} + 2\Big(\frac{q-1}{2}\Big)^{-s}\\ &+ \frac{2^{2+s}q(q^2-1)^{-s} + \frac{1}{2}(q-1)(q-\varepsilon)(q^2-\varepsilon q)^{-s} + \frac{1}{2}(q-1)(q-\varepsilon)(q^2+\epsilon q)^{-s}}{1-q^{1-s}}\end{aligned}$$

for places v of norm $q \geq 5$ where $\mathbf{H}$ is SL_2 or SU_2 and accordingly $\varepsilon = 1$ resp. -1. By a similar Taylor argument as before we obtain

$$\zeta_{\mathbf{H}(\mathcal{O}_v)}(s) = (1 - q^{1-s})^{-1}\big(1 + O_s(q^{1-2\Re s} + q^{-\Re s})\big)$$

and the meromorphic continuation to $\Re s > 1$ follows easily. The asymptotic formula follows as in the previous proof.

To analyze the Euler product further as we approach $\Re s = 1$, we continue with Taylor expansions, and for $\Re s > 1/2$ we write

$$1 - \frac{\varepsilon}{q^s} + \frac{2^{2+2s} - 1}{q^{2s-1}} = \Big(1 + \frac{\epsilon}{q^s}\Big)^{-1}\Big(1 - \frac{1}{q^{2s-1}}\Big)^{1-2^{2+s}}\Big(1 + O_s\big(q^{2-4\Re s} + q^{-2\Re s}\big)\Big).$$

If we write as before $\chi(\mathfrak{p}) = \varepsilon_v$ for a finite place $v = \mathfrak{p}$, we obtain

$$\zeta_{\mathbf{H}(\mathcal{O}_S)}(s) = \zeta(s)^r \frac{\zeta_k^S(s-1)}{L^S(s,\chi)}\zeta_k^S(2s-1)^{2^{2+s}-1}H(s)$$

(with $r = 0$ in the function field case and $r = [k : \mathbb{Q}]$ in the number field case) for $\Re s > 1$ where H is holomorphic and nonzero in $\Re s > 3/4$. As ζ_k has a pole at $s = 1$, the factor $\zeta_k(2s-1)^{2^{2+s}-1}$ and hence

$$\zeta_k^S(2s-1)^{2^{2+s}-1}$$

has a branch cut singularity at $s = 1$. More precisely, as $\zeta_k(s) = c(s-1)^{-1} + O(1)$ as $s \to 1$ for a constant $c \neq 0$, we have

$$\zeta_k(2s-1)^{2^{2+s}-1} = \frac{c^7}{128(s-1)^7} + \frac{c^7 \log 2}{16(s-1)^6}\log\Big(\frac{\frac{1}{2}c}{(s-1)}\Big) + O(|s-1|^6).$$

The factor $\zeta(s)^r/L^S(s,\chi)$ may change the exponent 7, but does not affect the type of the brunch cut singularity.

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