# PARTITIONS, FLAGS, TABLEAUX: COMBINATORIAL ASPECTS OF LATTICE ENUMERATION

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Enumerating sublattices of $\mathbb{Z}^n$ by their index is a classical problem with an elegant solution; cf. Theorem 1.1. In this note I argue that the simplest way to obtain it may not be the most interesting one: by refining the original enumeration problem and widening our point of view we discover not only new proofs but, more importantly, links to various circles of ideas from algebra, geometry, and combinatorics. Concretely, we will find connections with the Hermite and Smith normal forms of integer matrices, Gaussian multinomial coefficients, permutation statistics, Hall–Littlewood symmetric polynomials, Young tableaux, and Schubert varieties.

## 1. Enumerating lattices by index

By a *lattice* we mean a subgroup of finite index in $\mathbb{Z}^n$. One rather coarse way to enumerate lattices $\Lambda$ is by their *index* $|\mathbb{Z}^n : \Lambda|$ in $\mathbb{Z}^n$. I like to think of this invariant as the "inverse density" of the points in $\Lambda$ among all vectors in $\mathbb{Z}^n$. For $m \in \mathbb{N}$, let us write $a_m(\mathbb{Z}^n)$ for the (finite!) number of sublattices of $\mathbb{Z}^n$ of index $m$. The *zeta function of* $\mathbb{Z}^n$ turns the infinitely many numbers $(a_m(\mathbb{Z}^n))_{m\in\mathbb{N}}$ into a single object:

$$\zeta_{\mathbb{Z}^n}(s) := \sum_{m=1}^{\infty} a_m(\mathbb{Z}^n) m^{-s}. \tag{1.1}$$

Here, $s$ denotes a complex variable. We write $\zeta(s) = \sum_{m=1}^{\infty} m^{-s} = \zeta_{\mathbb{Z}}(s)$ for the Riemann zeta function.

**Theorem 1.1.** *For all $n \in \mathbb{N}$ we have*

$$\zeta_{\mathbb{Z}^n}(s) = \prod_{i=1}^{n} \zeta(s - i + 1). \tag{1.2}$$

This result has been (re-)proved by several people in several contexts. The oldest source I am aware of—at least for the ideas—goes back to Hermite [5]. In fact, Theorem 1.1 may be proven quickly using integer matrices in *Hermite normal form* (*HNF*): after a choice of $\mathbb{Z}$-basis for $\mathbb{Z}^n$, we may represent a lattice by an integral matrix which encodes in its columns, say, coordinates of generators. A unique such matrix is in HNF, the integral analogue of reduced echelon form for matrices over fields. The index of the lattice is just the determinant of the matrix; the $i$-th factor in (1.2) takes care of the contribution of the $i$-th column to this count.

**Example 1.2** ($n = 3$, $p = 5$)**.** The matrix

$$M_{\mathrm{HNF}} = \begin{pmatrix} 25 & 10 & 3 \\ & 125 & 100 \\ & & 125 \end{pmatrix} \in \mathrm{Mat}_3(\mathbb{Z})$$

is in Hermite normal form, representing a lattice $\Lambda \leqslant \mathbb{Z}^3$ of index $5^{2+3+3}$.

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The formula in Theorem 1.1 also occurs in work of Solomon on the integral representation theory of finite groups [9]. The monograph [7] contains no fewer than five proofs of this theorem, all group-theoretically motivated. Here we focus on ideas from algebraic combinatorics and geometry.

## 2. Enumerating lattices by partitions

Enumerating lattices $\Lambda$ simply by their index disregards the finer structure of the finite abelian group $\mathbb{Z}^n/\Lambda$. It is a direct product, indexed by the prime numbers, of finite abelian $p$-groups of the form

$$G_{p^\lambda} := \bigoplus_{i=1}^{n} C_{p^{\lambda_i}}, \tag{2.1}$$

where $\lambda = (\lambda_1, \ldots, \lambda_n)$ is a partition of at most $n$ parts and $C_{p^\ell}$ is the cyclic group of order $p^\ell$. This decomposition allows us to focus on lattices of $p$-power index, where $p$ is a prime number. In this case we have $\mathbb{Z}^n/\Lambda \cong G_{p^\lambda}$ for a single partition $\lambda$ which we call the *type* of the lattice $\Lambda$. Equation (2.1) implies the identity

$$a_m(\mathbb{Z}^n) = \prod_{p \text{ prime}} \left( \sum_{\lambda \vdash e(m,p)} a_{p,\lambda}(\mathbb{Z}^n) \right),$$

where $a_{p,\lambda}(\mathbb{Z}^n)$ is the number of lattices in $\mathbb{Z}^n$ of $p$-power index and of type $\lambda$, the product ranges over all prime numbers and the sum over all partitions $\lambda$ of the exponent $e(m,p)$ in the prime decomposition $m = \prod_{p \text{ prime}} p^{e(m,p)}$.

Assume from now on that $\Lambda$ has $p$-power index and type $\lambda$. The numbers $\left(p^{\lambda_1}, \ldots, p^{\lambda_n}\right)$ are exactly the *elementary divisors* of the $p$-group $G_{p^\lambda}$ or, equivalently, the diagonal entries of a matrix in *Smith normal form* (*SNF*) representing $\Lambda$. Note that the elementary divisors are intrinsic invariants of a lattice, whereas the HNF matrix representing it depends on a choice of basis for (or rather—as we will discuss in Section 3—flag in) $\mathbb{Z}^n$.

For a beautiful formula for the numbers $a_{p,\lambda}(\mathbb{Z}^n)$ we write the *dual* partition

$$\lambda' = (i_1^{(r_{i_1})}, \ldots, i_\ell^{(r_{i_\ell})}) = (\underbrace{i_1, \ldots, i_1}_{r_{i_1} \text{ times}}, \ldots, \underbrace{i_\ell, \ldots, i_\ell}_{r_{i_\ell} \text{ times}})$$

of $\lambda$, for a unique subset $I = \{i_1, \ldots, i_\ell\} \subseteq [n]$ of size $\ell$ and $r_I = (r_{i_1}, \ldots, r_{i_\ell}) \in \mathbb{N}^I$. I like to think of the set $I$ as the positions of the "jumps" in the partition $\lambda$, of heights recorded by the vector $r_I$. Then

$$a_{p,\lambda}(\mathbb{Z}^n) = \binom{n}{I}_{p^{-1}} p^{\sum_{i \in I} i(n-i)\, r_i}. \tag{2.2}$$

Here $\binom{n}{I}_Y = \binom{n}{i_\ell}_Y \binom{i_\ell}{i_{\ell-1}}_Y \cdots \binom{i_2}{i_1}_\ell$ is the $Y$- (or *Gaussian*) *multinomial coefficient*, generalizing the usual multinomial coefficient which we recover at $Y = 1$.[1] For a prime power $q$ and a finite field $K$ of cardinality $q$, the quantity $\binom{n}{I}_q$ is the number of *flags*

$$V_{i_1} < V_{i_2} < \cdots < V_{i_\ell} \leqslant K^n$$

comprising $i$-dimensional $K$-linear subspaces $V_i$ of $K$ for $i \in I$. We note that $i(n-i)$ is the dimension of the *Grassmannian variety* of $i$-dimensional subspaces of an $n$-dimensional vector space. Equation (2.2) may be proved using a result of Birkhoff on the numbers of subgroups of finite abelian $p$-groups of given type [2].

[1] If, like me, you are wary of a polynomial in the *inverse* of $p$ occuring in a formula for a natural number like $a_{p,\lambda}(\mathbb{Z}^n)$, you may want to check that the subsequent $p$-power is powerful enough to clear any denominators, so all is well.

Birkhoff's formula (2.2) suggests a second proof of Theorem 1.1. For this, we exploit the fact that it presents $a_{p,\lambda}(\mathbb{Z}^n)$ as the product of

(1) a polynomial in $p^{-1}$, depending only on one of finitely many $I \subseteq [n]$ and
(2) a $p$-power whose exponent is linear in the natural parameters $r_i \in \mathbb{N}$.

Products of finitely many geometric progressions do the job of summing over these "log-linear" terms: indeed, purely formally we have

$$\sum_{r_I \in \mathbb{N}^I} X_{i_1}^{r_{i_1}} \dots X_{i_\ell}^{r_{i_\ell}} = \prod_{i \in I} \frac{X_i}{1 - X_i}. \tag{2.3}$$

From this it is not hard to deduce that

$$\zeta_{\mathbb{Z}^n}(s) = \prod_{p \text{ prime}} \left( \sum_{I \subseteq [n]} \binom{n}{I}_{p^{-1}} \prod_{i \in I} \frac{p^{i(n-i)-is}}{1 - p^{i(n-i)-is}} \right). \tag{2.4}$$

2.1. **Igusa zeta functions and permutation statistics.** The factors of the *Euler product* (2.4) are instances of *Igusa functions* of degree $n$, defined as follows:

$$\mathsf{I}_n(Y; X_1, \dots, X_n) = \sum_{I \subseteq [n]} \binom{n}{I}_{Y} \prod_{i \in I} \frac{X_i}{1 - X_i} \in \mathbb{Z}[Y](X_1, \dots, X_n). \tag{2.5}$$

Originally introduced by Igusa [6], these functions are a perfect fit for enumeration problems involving $p$-adic lattices by invariants that depend on the types of the lattices in a log-linear fashion. For references and generalizations, see [3].

Interesting connections with algebraic combinatorics emerge when we bring the sum (2.5) on a common denominator:

$$\mathsf{I}_n(Y; \boldsymbol{X}) = \frac{\sum_{w \in S_n} Y^{\ell(w)} \prod_{j \in \mathrm{Des}(w)} X_j}{\prod_{i=1}^n (1 - X_i)}. \tag{2.6}$$

Here, $S_n$ is the symmetric group on $[n]$, a Coxeter group with *length function* $\ell$, and $\mathrm{Des}(w)$ is the *descent set* of $w \in S_n$. Equation (2.6) reflects the well-known fact that

$$\binom{n}{I}_Y = \sum_{w \in S_n \colon \mathrm{Des}(w) \subseteq I} Y^{\ell(w)}.$$

In light of (2.6) it is remarkable that the numerator of $\mathsf{I}_n$ gets entirely cancelled out after the substitutions $Y = p^{-1}$, $X_i = p^{i(n-i)-is}$. Indeed, combining Theorem 1.1 and (2.4) yields

$$\frac{1}{\prod_{i=1}^n (1 - p^{i-1-s})} = \sum_{I \subseteq [n]} \binom{n}{I}_{p^{-1}} \prod_{i \in I} \frac{p^{i(n-i)-is}}{1 - p^{i(n-i)-is}}.$$

2.2. **An excursion: $p$-adic lattices and affine buildings.** Let $\mathbb{Z}_p$ be the ring of $p$-adic integers and $\mathbb{Q}_p$ its field of fractions. Affine buildings associated with the groups $\mathrm{GL}_n(\mathbb{Q}_p)$ offer a coordinate-free way to parametrize lattices of $p$-power index in $\mathbb{Z}^n$. One way to define these buildings is as abstract simplicial complexes on the set of *homothety classes* of lattices in $\mathbb{Q}_p^n$, by defining a suitable incidence relation. The resulting $(n-1)$-dimensional simplicial complex may be thought of as glued together from subcomplexes called *apartments*. The latter are Coxeter complexes of type $\widetilde{A}_{n-1}$. Geometrically, they are Euclidean spaces, tessellated by regular $(n-1)$-dimensional simplices. Algebraically, they are obtained from lattices which are generated by $p$-power multiples of the members of a given (unordered) basis of $\mathbb{Q}_p^n$.

A glimpse at the glass roof of the Great Court of the British Museum (Figure 2.1) gives an idea how such an apartment looks like for $n = 3$. We see lattices wherever six of the triangular glass panels meet at a vertex. Note that $6 = \binom{3}{1} + \binom{3}{2}$ is the total number of non-trivial subspaces of $K^3$, where $K$ is any field, that can be generated from the members of a fixed $K$-basis of $K^3$.

FIGURE 2.1. Lattices spanning the roof of the Great Court of the British Museum, London.

In (any apartment containing) Sir Foster's glass roof, exactly two panels meet at any given segment connecting incident vertices. In the full affine building associated with $\mathrm{GL}_3(\mathbb{Q}_p)$, each such segment is part of $p+1$ such panels. Geometrically, the "reduction" $p \mapsto 1$ corresponds to the retraction of the affine building to the apartment we started from. The book [4] not only explains the realization of affine buildings by $p$-adic lattices in an inviting way. It also illustrates the rich combinatorial structure of affine buildings with various pictures. *The Buildings Gallery* ([1]) offers appealing interactive computer visualizations.

The language of affine buildings affords another view on Birkhoff's formula (2.2). Indeed, the group $\mathrm{GL}_n(\mathbb{Z}_p)$ acts in a natural way on the vertices of the affine building for $\mathrm{GL}_n(\mathbb{Q}_p)$. This action is transitive on the sets of vertices comprising lattices of given type $\lambda$. By the orbit stabilizer theorem, the number of such lattices is thus the index of the stabilizer of any one of these lattices. An explicit formula for one such stabilizer, from which the relevant index in $\mathrm{GL}_n(\mathbb{Z}_p)$ may be readily read off, is given in [10, p. 1203].

**Example 2.1** (Example 1.2 revisited)**.** The SNF matrix representing the lattice $\Lambda$ from Example 1.2 is

$$M_{\mathrm{SNF}} = \begin{pmatrix} 5^5 & & \\ & 5^3 & \\ & & 1 \end{pmatrix} \in \mathrm{Mat}_3(\mathbb{Z}),$$

yielding $I = \{1,2\} \subseteq [3]$ and $r_I = (2,3)$. It is thus one of

$$\binom{3}{1,2}_{5^{-1}} 5^{3\cdot 2(3-2)+2\cdot 1(3-1)} = (1 + 2\cdot 5^{-1} + 2\cdot 5^{-2} + 5^{-3})\, 5^{10} = 14.531.250$$

lattices of 5-power index and type $\lambda = (5,3,0)$. Note that $\lambda$ does not encode the powers of 5 on the diagonal of the HNF matrix $M_{\mathrm{HNF}}$.

## 3. Enumerating lattices by tableaux

We now take on the challenge to enumerate lattices *simultaneously* by their Hermite and Smith normal forms. In coordinate-free language, we enumerate lattices in $\mathbb{Z}^n$—without loss of generality of prime-power index—according to the types of their intersections with the parts of a fixed complete flag

$$\mathbb{Z} < \mathbb{Z}^2 < \cdots < \mathbb{Z}^{n-1} < \mathbb{Z}^n; \tag{3.1}$$

see Figure 3.1 for an example.

Figure 3.1. A flag over the Lac Leman near Lausanne.

We associate, in other words, with a lattice $\Lambda$ not just one partition $\lambda$ as in Section 2, but $n$ partitions $\lambda^{(i)} = (\lambda_1^{(i)}, \ldots, \lambda_i^{(i)})$, one for each $i \in [n]$, each of at most $i$ non-zero parts. They record the types of the rank-$i$ lattices $\Lambda \cap \mathbb{Z}^i$, i.e. the isomorphism types of the finite abelian $p$-groups $G_{p^{\lambda^{(i)}}} \cong \mathbb{Z}^i/(\Lambda \cap \mathbb{Z}^i)$. (Note that $\lambda^{(n)} = \lambda$, the type of $\Lambda$ itself.) These partitions do not differ too much from one to the next, in a precise sense: the successive differences of their Young diagrams form *horizontal strips*. In other words, the sequence $\lambda^{\bullet}(\Lambda) = (\lambda^{(1)}, \ldots, \lambda^{(n)})$ of partitions encodes the same data as a *semistandard Young tableau* $T$ of shape $\lambda$ with labels from $[n]$. Write $\mathrm{SSYT}_n$ for the set of all such tableaux. We call a tableau *reduced* if it has no repeated columns and write $\mathrm{rSSYT}_n$ for the finite (!) set of reduced tableaux with labels from $[n]$.

This allows us to meet the above challenge: the lattices' elementary divisors are recorded in $\lambda^{(n)}$, its Hermite normal form (with respect to any basis compatible with the fixed flag) is encoded in the (reverse) *weight* of $T$, i.e. the multiplicities of the numbers $n, \ldots, 2, 1$ as labels of $T$, in this order. Note, however, that the tableau $T$ associated with a lattice $\Lambda$ records much more than these two sets of data: in general, there are numerous tableaux with the same weight.

How many lattices $\Lambda$ of $p$-power index give rise to a tableau $T \in \mathrm{SSYT}_n$ in this way? The answer takes the form

$$\Phi_T(p^{-1})\, p^{\sum_{C\in T} D_n(C)}, \tag{3.2}$$

where $\Phi_T(Y)$ is the *leg polynomial* of $T$—a coefficient of the *Hall–Littlewood polymial* associated with the partition $\lambda$—and $C$ ranges over the columns of $T$, viewed as subsets of $[n]$; see [8, Thm. 4.7]. The quantity $D_n(C) = \left(\sum_{i\in[n]\setminus C} i\right) - \binom{n-\#C+1}{2}$ is the dimension of the *Schubert variety* associated with $C$.

**Example 3.1** (Example 1.2 re$^2$visited)**.** We fix an ordered basis $(e_3, e_2, e_1)$ (sic!) and the induced flag $\langle e_1\rangle < \langle e_1, e_2\rangle < \langle e_1, e_2, e_3\rangle$. We find that, with respect to this basis, $\lambda^\bullet(\Lambda) = ((), (3), (4,2), (5,3,0))$. Equivalently, the tableau in $\mathrm{SSYT}_3$ associated with $\Lambda$ is

$$T = \begin{array}{|c|c|c|c|c|}\hline 1&1&1&2&3\\\hline 2&2&3 \\\cline{1-3}\end{array}$$

with weight $(2,3,3)$ and leg polynomial $\Phi_T(Y) = (1-Y)^3$. The lattice $\Lambda$ is one of

$$\Phi_T(p^{-1})\, p^{\sum_{C\in T}\left(\left(\sum_{i\in[n]\setminus C} i\right)-\binom{n-\#C+1}{2}\right)} = (1-5^{-1})^3\,5^6 = 8.000$$

lattices giving rise to the tableau $T$.

I invite you to draw a few parallels between (3.2) and (2.2). Both formulae are products of a polynomial in $p^{-1}$ and a $p$-power whose exponent depends linearly on the multiplicities of parts taken from a finite (!) combinatorial structure: parts of the dual partition of $\lambda$ in the former case, sets of labels in the columns of the tableau $T$ in the latter. The coefficients of these linear forms are dimensions of algebraic varieties: Grassmannians in the former case, Schubert varieties in the latter.

That Schubert varieties occur in this context I find both surprising and natural. Indeed, Schubert varieties parametrize subspaces of a vector space by the (co-)dimensions of their intersections with a flag of reference. In the affine analogue we parametrize lattices by the types of their intersections with a flag of reference.

The Igusa function (2.5) funnels the infiniteness of the set of lattice types into finitely many subsets indexed by the subsets of $[n]$, each easily enumerable by products of geometric progressions. Likewise, the *Hall–Littlewood–Schubert series*

$$\mathsf{HLS}_n\,(Y, \boldsymbol{X}) = \sum_{T\in \mathrm{rSSYT}_n} \Phi_T(Y) \prod_{C\in T} \frac{X_C}{1-X_C} \in \mathbb{Z}[Y]\,(\boldsymbol{X})\,. \tag{3.3}$$

organizes the infinitude of tableaux by the finitely many reduced tableaux. In $\boldsymbol{X} = (X_C)_{\varnothing\neq C\subseteq[n]}$ we comprise $2^n-1$ variables, one for each "column-type", viz. set of labels in $[n]$ occurring in a column of a tableau $T \in \mathrm{SSYT}_n$. Monomials in these variables record multiplicities of tableaux columns with the same labels. Products of geometric progressions enumerate the fibres of the natural surjection $\mathrm{SSYT}_n \to \mathrm{rSSYT}_n$ that removes repetitions of columns. We saw the same idea at work in the Igusa function (2.5); see also (2.3).

**Example 3.2** (Example 1.2 re$^3$visited)**.** Removing the first column of $T$ in (3.1) yields a reduced tableau $T'$, with leg polynomial $\Phi_{T'}(Y) = (1-Y)^3$. The (non-reduced) tableaux $T$ contributes to $\mathsf{HLS}_3(Y, \boldsymbol{X})$ the term

$$(1-Y)^3\; X_{\{1,2\}}^2 X_{\{1,3\}} X_{\{2\}} X_{\{3\}}.$$

It is subsumed as a summend in the product of geometric progressions indexed by the reduced tableaux $T'$, viewed as an infinite sum.

With a variable $X_C$ for each non-empty label set $C \subseteq [n]$, there is plenty of room for coarsenings of $\mathsf{HLS}_n$, i.e. monomial substitutions of the variables, to obtain solutions to interesting (lattice) counting problems. The enumeration of lattices by Smith and Hermite normal forms simultaneously is just one of them; see [8, Thm. D]. Other applications include Hecke series associated with symplectic groups and representation zeta functions of integral quiver representations; see [8] for details.

Let's wrap up. We associate with a lattice $\Lambda \leqslant \mathbb{Z}^n$ of prime-power index three increasingly fine-grained invariants. The finest of these is a semistandard Young tableau $T \in \mathrm{SSYT}_n$, measuring how $\Lambda$ intersects with the members of a complete flag within $\mathbb{Z}^n$. Forgetting the labels in the tableau $T$ leaves us, secondly, with the Young diagram of a partition $\lambda$, yielding the type of the finite abelian $p$-group $\mathbb{Z}^n/\Lambda$. The number partitioned by $\lambda$ yields, finally, the index of $\Lambda$ in $\mathbb{Z}^n$. For each of the three invariants we identified combinatorial structures tailor-made to enumerate lattices accordingly: the ($p$-local factor of the) zeta function $\zeta_{\mathbb{Z}^n}$ in (1.1) for the index, the Igusa function $\mathsf{I}_n$ in (2.5) for the type, the Hall–Littlewood–Schubert series $\mathsf{HLS}_n$ in (3.3) for the tableau. With an exponential number of variables, $\mathsf{HLS}_n$ is the richest of these structures, allowing for various monomial substitutions solving numerous related lattice enumeration problems, many of which, no doubt, are yet to be discovered.

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